

COSET CLOSURE OF A CIRCULANT S-RING AND SCHURITY PROBLEM

SERGEI EVDOKIMOV AND ILYA PONOMARENKO

ABSTRACT. Let G be a finite group. There is a natural Galois correspondence between the permutation groups containing G as a regular subgroup, and the Schur rings (S-rings) over G . The problem we deal with in the paper, is to characterize those S-rings that are closed under this correspondence, when the group G is cyclic (the schurity problem for circulant S-rings). It is proved that up to a natural reduction, the characteristic property of such an S-ring is to be a certain algebraic fusion of its coset closure introduced and studied in the paper. Basing on this characterization we show that the schurity problem is equivalent to the consistency of a modular linear system associated with a circulant S-ring under consideration. As a byproduct we show that a circulant S-ring is Galois closed if and only if so is its dual.

1. INTRODUCTION

A *Schur ring* or *S-ring* over a finite group G can be defined as a subring of the group ring $\mathbb{Z}G$ that is a free \mathbb{Z} -module spanned by a partition of G closed under taking inverse and containing $\{1_G\}$ as a class (see Section 2 for details). It is well known that there is a Galois correspondence¹ between the permutation groups on G that contain the regular group G_{right} , and the S-rings over G :

$$(1) \quad \{\Gamma \leq \text{Sym}(G) : \Gamma \geq G_{right}\} \rightleftharpoons \{\mathcal{A} \leq \mathbb{Z}G : \mathcal{A} \text{ is an S-ring over } G\}.$$

More precisely, the " \rightarrow " mapping is given by taking the partition of G into the orbits of the stabilizer of 1_G in Γ , whereas the " \leftarrow " mapping is given by taking the automorphism group of the colored Cayley graph corresponding to the partition of G associated with \mathcal{A} . The Galois closed objects are called 2-closed groups and *schurian* S-rings, respectively. The schurity problem consists in finding an inner characterization of schurian S-rings.

The theory of S-rings was initiated by I. Schur (1933) and later developed by H. Wielandt [17] and his followers. The starting point for Schur was the Burnside theorem stating that any primitive permutation group containing a regular cyclic p -group of composite order, is 2-transitive. Using the S-ring method introduced by him, Schur generalized this theorem to an arbitrary finite cyclic group G (cf. Theorem 5.1). To some extent this explains the fact that "Schur had conjectured for a long time that every S-ring over G is determined by a suitable permutation group" [18, p.54]. This statement had been known as the Schur-Klin conjecture up to 2001, when the first examples of circulant (i.e. over a cyclic group) S-rings were constructed in [3] by the authors. A recent result in [10] shows that schurian

The second author was partially supported by RFFI Grant 14-01-00156.

¹We recall that a Galois correspondence between two posets consists of two mappings reversing the orders such that both superpositions are closure operators.

circulant S-rings are relatively rare. In this paper we provide a solution to the schurity problem for circulant S-rings.

The non-schurian examples of S-rings were constructed using the operation of *generalized wreath product* introduced in [3] (and independently in [13] under the name "wedge product"). This is not surprising due to the seminal Leung-Man theorem according to which any circulant S-ring can be constructed from S-rings of rank 2 and cyclotomic S-rings by means of two operations: tensor product and generalized wreath product [13]. Here under a *cyclotomic* S-ring \mathcal{A} we mean the ring of all K -invariant elements of $\mathbb{Z}G$ where K is a subgroup of $\text{Aut}(G)$:

$$(2) \quad \mathcal{A} = (\mathbb{Z}G)^K.$$

The Leung-Man theorem reduces the schurity problem for circulant S-rings to finding a criterion for the schurity of the generalized wreath product. Such a criterion, based on a generalization of the Leung-Man theory (see [4]), was obtained in paper [9] where the generalized wreath product of permutation groups was introduced and studied. All these results form a background to prove the main results of the paper (see Sections 2–5).

Let \mathcal{A} be a circulant S-ring. Suppose that among the "bricks" in the Leung-Man decomposition of \mathcal{A} , there is a non-cyclotomic S-ring. Then this ring is of rank 2, its underlying group has composite order and it is Cayley isomorphic to the restriction of \mathcal{A} to one of its sections. Moreover, as it was proved in [9] the S-ring \mathcal{A} has a quite a rigid structure that enables us to control the schurity of \mathcal{A} . This provides a reduction of the schurity problem to the case when \mathcal{A} has no rank 2 section of composite order. The S-rings satisfying the latter property are *quasidense* in sense of paper [10] (Theorem 5.5). Thus without loss of generality we concentrate on the schurity problem for quasidense S-rings.

Our first step is to represent the schurian closure $\text{Sch}(\mathcal{A})$ of a quasidense circulant S-ring \mathcal{A} in a regular form (Theorem 1.2). The idea here is to replace the ring \mathcal{A} by a simpler one keeping the structure of its Leung-Man decomposition. The simplification is achieved by changing each "brick" for a group ring. This leads to the class of *coset* S-rings, i.e. ones for which any class of the corresponding partition of the group G is a coset of a subgroup in G . It appears that this class is closed under restriction to a section, tensor and generalized wreath products, and consists of schurian quasidense S-rings (Theorems 8.4 and 9.1). The regular form of $\text{Sch}(\mathcal{A})$ we want to come, will be defined by means of the following concept.

Definition 1.1. *The coset closure of a quasidense circulant S-ring \mathcal{A} is the intersection \mathcal{A}_0 of all coset S-rings over G that contain \mathcal{A} .*

The coset closure of any quasidense circulant S-ring is a coset S-ring (Theorem 8.5). Now, to clarify how to represent the schurian closure of \mathcal{A} via its coset closure, suppose that the group G is of prime order. In this case it is well known that the S-ring \mathcal{A} is of the form (2), and, moreover, $\mathcal{A}_0 = \mathbb{Z}G$. In particular, \mathcal{A} is schurian and any automorphism of G induces a similarity of \mathcal{A}_0 .² Furthermore, if the automorphism belongs to the group K , the similarity is identical on \mathcal{A} . Thus

$$\mathcal{A} = (\mathbb{Z}G)^K = (\mathcal{A}_0)^{\Phi_0}$$

²Under a similarity of an S-ring \mathcal{A} we mean a ring isomorphism of it that respects the partition of G corresponding to \mathcal{A} , see Subsection 3.1.

where $\Phi_0 = \Phi_0(\mathcal{A})$ is the group of all similarities of \mathcal{A}_0 that are identical on \mathcal{A} . It appears that this idea works for any quasidense S-ring \mathcal{A} .

Theorem 1.2. *Let \mathcal{A} be a quasidense circulant S-ring. Then*

$$\text{Sch}(\mathcal{A}) = (\mathcal{A}_0)^{\Phi_0}.$$

In particular, \mathcal{A} is schurian if and only if $\mathcal{A} = (\mathcal{A}_0)^{\Phi_0}$.

Theorem 1.2 gives a necessary and sufficient condition for an S-ring to be schurian. This condition being a satisfactory from the theoretical point of view, is hardly an inner characterization. To obtain the latter, we prove Theorem 1.3 below. Let us discuss briefly the idea behind it.

One of the key properties of coset S-rings that is used in the proof of Theorem 1.2, is that every similarity of any such ring is induced by isomorphism (Theorem 9.1). This fact also shows that in the schurian case the set of all isomorphisms of \mathcal{A}_0 that induce similarities belonging to Φ_0 , forms a permutation group the associated S-ring of which coincides with \mathcal{A} . In general, this is not true. A rough reason for this can be explained as follows. Set

$$(3) \quad \mathfrak{S}_0 = \{S \in \mathfrak{S}(\mathcal{A}_0) : (\mathcal{A}_0)_S = \mathbb{Z}S\}$$

where $\mathfrak{S}(\mathcal{A}_0)$ is the set of all \mathcal{A}_0 -sections and $(\mathcal{A}_0)_S$ is the restriction of \mathcal{A}_0 to S . Then in the schurian case every S-ring \mathcal{A}_S with $S \in \mathfrak{S}_0$, must be cyclotomic, whereas in general this condition does not necessarily hold. However, if even all the S-rings \mathcal{A}_S are cyclotomic, one still might find a section S for which $\mathcal{A}_S \neq \text{Sch}(\mathcal{A})_S$. These two reasons are controlled respectively by conditions (1) and (2) of Theorem 1.3.

It should be mentioned that the proof of the fact that the circulant S-rings constructed in [3] are non-schurian, was based on studying the relationship between their cyclotomic sections. More careful analysis can be found in [5] where the isomorphism problem for circulant graphs was solved. In that paper the authors introduce and study the notion of projective equivalence on the sections of a circulant S-ring (this notion is similar to one used in the lattice theory). It appears that the class \mathfrak{S}_0 defined in (3) is closed under the projective equivalence and taking subsections (Corollary 10.10). Moreover,

$$\mathfrak{S}(\mathcal{A}_0) = \mathfrak{S}(\mathcal{A})$$

(statement (1) of Theorem 10.11).

To formulate Theorem 1.3 we need additional notation. For $S \in \mathfrak{S}(\mathcal{A})$ denote by $\text{Aut}_{\mathcal{A}}(S)$ the subgroup of $\text{Aut}(S)$ that consists of all Cayley automorphisms of the S-ring \mathcal{A}_S . A family

$$\Sigma = \{\sigma_S\}_{S \in \mathfrak{S}_0}$$

is called a *multiplier* of \mathcal{A} if for any sections $S, T \in \mathfrak{S}_0$ such that T is projectively equivalent to a subsection of S the automorphisms $\sigma_T \in \text{Aut}(T)$ and $\sigma_S \in \text{Aut}(S)$ are induced by raising to the same power³. The set of all multipliers of \mathcal{A} forms a subgroup of the direct product $\prod_{S \in \mathfrak{S}_0} \text{Aut}_{\mathcal{A}}(S)$ that is denoted by $\text{Mult}(\mathcal{A})$.

³We recall that any automorphism of a finite cyclic group is induced by raising to a power coprime to the order of this group.

Theorem 1.3. *A quasidense circulant S-ring \mathcal{A} is schurian if and only if the following two conditions are satisfied for all $S \in \mathfrak{S}_0$:*

- (1) *the S-ring \mathcal{A}_S is cyclotomic,*
- (2) *the restriction homomorphism from $\text{Mult}(\mathcal{A})$ to $\text{Aut}_{\mathcal{A}}(S)$ is surjective.*

By Theorem 10.5 the class \mathfrak{S}_0 consists of all \mathcal{A} -sections S such that each Sylow subgroup of S (treated as a section of G) is projectively equivalent to a subsection of a principal \mathcal{A} -section. Thus in contrast to Theorem 1.2, Theorem 1.3 gives a necessary and sufficient condition for an S-ring \mathcal{A} to be schurian in terms of \mathcal{A} itself rather than of its coset closure \mathcal{A}_0 . It should be remarked that in general the class \mathfrak{S}_0 may contain non-cyclotomic sections. However, we do not know whether condition (1) in Theorem 1.3 is implied by condition (2).

In fact, the starting point of this paper was the following question: is the property of an S-ring "to be schurian" preserved under taking the dual. The following theorem deduced in Section 13 from Theorem 1.3, answers this question in the positive.

Theorem 1.4. *A circulant S-ring is schurian if and only if so is the S-ring dual to it.*

We would like to reformulate Theorem 1.3 in the number theoretical language. In what follows we assume that condition (1) of that theorem is satisfied. To make condition (2) more clear let us fix a section $S_0 \in \mathfrak{S}_0$ and an integer b coprime to $n_{S_0} = |S_0|$ for which the mapping $s \mapsto s^b$, $s \in S_0$, belongs to $\text{Aut}_{\mathcal{A}}(S_0)$. Let us consider the following system of linear equations in integer variables x_S , $S \in \mathfrak{S}_0$:

$$(4) \quad \begin{cases} x_S \equiv x_T \pmod{n_T}, \\ x_{S_0} \equiv b \pmod{n_{S_0}} \end{cases}$$

where S and T run over \mathfrak{S}_0 and the section T is projectively equivalent to a subsection of S . We are interested only in the solutions of this system that satisfy the additional condition

$$(5) \quad (x_S, n_S) = 1 \quad \text{for all } S \in \mathfrak{S}_0.$$

Every such solution produces the family $\Sigma = \{\sigma_S\}$ where σ_S is the automorphism of the group S taking s to s^{x_S} . Moreover, the equations in the first line of (4) guarantee that if a section T is projectively equivalent to a subsection of S , then the automorphisms $\sigma_T \in \text{Aut}(T)$ and $\sigma_S \in \text{Aut}(S)$ are induced by raising to the same power. Therefore,

$$\Sigma \in \text{Mult}(\mathcal{A}).$$

Conversely, it is easily seen that given $S_0 \in \mathfrak{S}$ every multiplier of \mathcal{A} produces a solution of system (4) for the corresponding b . Finally, the consistency of this system for all S_0 and all possible b is equivalent to the surjectivity of the restriction homomorphism from $\text{Mult}(\mathcal{A})$ to $\text{Aut}_{\mathcal{A}}(S_0)$ for all S_0 . Thus we come to the following corollary of Theorem 1.3.

Corollary 1.5. *Let \mathcal{A} be a quasidense circulant S-ring such that for any section $S \in \mathfrak{S}_0$, the S-ring \mathcal{A}_S is cyclotomic. Then \mathcal{A} is schurian if and only if system (4) has a solution satisfying (5) for all possible S_0 and b .*

Corollary 1.5 reduces the schurity problem for circulant S-rings to solving modular linear system (4) under restriction (5). One possible way to solve this system is to represent the group $\prod_S \text{Aut}_{\mathcal{A}}(S)$ as a permutation group on the disjoint union of the sections S . Then every equation in the first line of (4) defines a subgroup of that group the index of which is at most n^2 . Therefore the set of solutions can be found by a standard permutation group technique, see [14, p. 144] for details.

Concerning permutation groups we refer to [2]. For the reader convenience an extended background on the S-ring theory including new material, is given in Sections 2–5. In Sections 6 and 7 we study liftings of generalized wreath products in the non-dense and dense cases respectively. The theory of coset S-rings is developed in Sections 8 and 9, and culminates in Theorem 9.1 showing that these S-rings are schurian and separable. The coset closure and multipliers of a quasidense S-ring are introduced and studied in Sections 10 and 11 respectively. Theorems 1.2 and 1.3 are proved in Section 12. In the final Section 13 we prove Theorem 1.4.

Notation. As usual by \mathbb{Z} we denote the ring of rational integers.

The set of all right cosets of a subgroup H in a group G is denoted by G/H . For a set $X \subset G$ we put $X/H = \{Y \in G/H : Y \subset X\}$.

The subgroup of G generated by a set $X \subset G$ is denoted by $\langle X \rangle$; we also set $\text{rad}(X) = \{g \in G : gX = Xg = X\}$.

For a prime p a Sylow p -subgroup of G is denoted by G_p .

For a section $S = U/L$ of G the quotient epimorphism from U onto S is denoted by π_S .

For a bijection $f : G \rightarrow G'$ and a set $X \subset 2^G$ or an element $X \in 2^G$, the induced bijection from X onto X^f is denoted by f^X .

The group of all permutations of G is denoted by $\text{Sym}(G)$.

For a set $\Delta \subset \text{Sym}(G)$ and a section S of G we set

$$\Delta^S = \{f^S : f \in \Delta, S^f = S\}.$$

The subgroup of $\text{Sym}(G)$ induced by right multiplications of G is denoted by G_{right} .

For $X, Y \in G/H$ we set $G_{X \rightarrow Y} = \{f^X : f \in G_{\text{right}}, X^f = Y\}$.

The holomorph $\text{Hol}(G)$ is identified with the subgroup of $\text{Sym}(G)$ generated by G_{right} and $\text{Aut}(G)$.

The orbit set of a group $\Gamma \leq \text{Sym}(G)$ is denoted by $\text{Orb}(\Gamma) = \text{Orb}(\Gamma, G)$.

We write $\Gamma \approx_2 \Gamma'$ if groups $\Gamma, \Gamma' \leq \text{Sym}(G)$ are 2-equivalent, i.e. have the same orbits in the coordinate-wise action on $G \times G$.

For a positive integer n , the cyclic group the elements of which are integers modulo n , is denoted by \mathbb{Z}_n .

For an automorphism σ of a cyclic group G of order n the unique element $m \in \mathbb{Z}_n$ for which $x^\sigma = x^m$, $x \in G$, is denoted by $m(\sigma)$.

2. S-RINGS

In what follows we use the notation and terminology from paper [9] where a part of the material is contained. Concerning the basic S-ring theory and duality theory we also refer to [1, Ch.2.6] and [17, Ch.4], respectively.

2.1. Definitions and basic facts. Let G be a finite group with identity 1_G . A subring \mathcal{A} of the group ring $\mathbb{Z}G$ is called a *Schur ring* (*S-ring*, for short) over G if there exists a partition $\mathcal{S}(\mathcal{A})$ of G such that

- (1) $\{1_G\} \in \mathcal{S}(\mathcal{A})$,
- (2) $X \in \mathcal{S}(\mathcal{A}) \implies X^{-1} \in \mathcal{S}(\mathcal{A})$,
- (3) $\mathcal{A} = \text{span}\{\sum_{x \in X} x : X \in \mathcal{S}(\mathcal{A})\}$.

Two S-rings \mathcal{A} and \mathcal{A}' are called *Cayley isomorphic* if there exists a ring isomorphism from \mathcal{A} onto \mathcal{A}' that is induced by isomorphism of underlying groups; the latter is called *Cayley isomorphism* from \mathcal{A} onto \mathcal{A}' . Obviously, $\mathcal{S}(\mathcal{A})^f = \mathcal{S}(\mathcal{A}')$ for any such isomorphism f .

The elements of $\mathcal{S}(\mathcal{A})$ and the number $\text{rk}(\mathcal{A}) = |\mathcal{S}(\mathcal{A})|$ are called respectively the *basic sets* and the *rank* of the S-ring \mathcal{A} . Any union of basic sets is called an *\mathcal{A} -subset of G* or *\mathcal{A} -set*. The set of all of them is closed with respect to taking inverse and product, and forms a lattice with respect to inclusion. Given an \mathcal{A} -set X the submodule of \mathcal{A} spanned by the set

$$\mathcal{S}(\mathcal{A})_X = \{Y \in \mathcal{S}(\mathcal{A}) : Y \subset X\}$$

is denoted by \mathcal{A}_X .

A subgroup of G that is an \mathcal{A} -set is called *\mathcal{A} -subgroup* of G or *\mathcal{A} -group*; the set of all of them is denoted by $\mathcal{G}(\mathcal{A})$. With each \mathcal{A} -set X one can naturally associate two \mathcal{A} -groups, namely $\langle X \rangle$ and $\text{rad}(X)$ (see Notation). The S-ring \mathcal{A} is called *dense* if every subgroup of G is an \mathcal{A} -group, and *primitive* if the only \mathcal{A} -subgroups are 1 and G .

A section $S = U/L$ of the group G is called a section of \mathcal{A} or an *\mathcal{A} -section*, if both U and L are \mathcal{A} -groups. In this case we also say that \mathcal{A} contains S . The module

$$\mathcal{A}_S = \text{span}\{\pi_S(X) : X \in \mathcal{S}(\mathcal{A})_U\}$$

is an S-ring over the group S , the basic sets of which are exactly the sets in the right-hand side of the formula. A section S is of rank 2 (resp. primitive, cyclotomic) if so is the S-ring \mathcal{A}_S . The set of all (resp. all cyclotomic) \mathcal{A} -sections is denoted by $\mathfrak{S}(\mathcal{A})$ (resp. $\mathfrak{S}_{\text{cyc}}(\mathcal{A})$). Set

$$\mathfrak{S}_{\text{prin}}(\mathcal{A}) = \{\langle X \rangle / \text{rad}(X) : X \in \mathcal{S}(\mathcal{A})\}.$$

Any element of this set is called a *principal \mathcal{A} -section*.

The partial order on the set of all S-rings over G that is induced by inclusion, is denoted by “ \leq ”. Thus $\mathcal{A}_1 \leq \mathcal{A}_2$ if and only if any basic set of \mathcal{A}_1 is a union of basic sets of \mathcal{A}_2 (in this case we say that \mathcal{A}_2 is an extension of \mathcal{A}_1). The least and greatest elements are

$$\text{span}\{1, \sum_{x \in G} x\} \quad \text{and} \quad \mathbb{Z}G$$

respectively. Next, the module $\mathcal{A}_1 \cap \mathcal{A}_2$ is an S-ring, the basic sets of which form the finest partition of G that is coarser than both $\mathcal{S}(\mathcal{A}_1)$ and $\mathcal{S}(\mathcal{A}_2)$. It follows that the set of all S-rings over G forms a lattice in which the meet (resp. join) of \mathcal{A}_1 and \mathcal{A}_2 coincides with $\mathcal{A}_1 \cap \mathcal{A}_2$ (resp. with the intersection of all S-rings over G that contain both \mathcal{A}_1 and \mathcal{A}_2). It is easily seen that

$$(6) \quad (\mathcal{A}_1 \cap \mathcal{A}_2)_S = (\mathcal{A}_1)_S \cap (\mathcal{A}_2)_S$$

for any section $S \in \mathfrak{S}(\mathcal{A}_1) \cap \mathfrak{S}(\mathcal{A}_2)$.

Let $K \leq \text{Aut}(G)$ be a group of Cayley isomorphisms of the S-ring \mathcal{A} . Then the set \mathcal{A}^K of the elements in \mathcal{A} left fixed under the induced action of K on $\mathbb{Z}G$, forms an S-ring over G . Any such ring with $\mathcal{A} = \mathbb{Z}G$, is called *cyclotomic* and is denoted by $\text{Cyc}(K, G)$. The classes of the corresponding partition of G are exactly the orbits of the group K .

If \mathcal{A}_1 and \mathcal{A}_2 are S-rings over groups G_1 and G_2 respectively, then the subring $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ of the ring $\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 = \mathbb{Z}G$ where $G = G_1 \times G_2$, is an S-ring over the group G with

$$\mathcal{S}(\mathcal{A}) = \{X_1 \times X_2 : X_1 \in \mathcal{S}(\mathcal{A}_1), X_2 \in \mathcal{S}(\mathcal{A}_2)\}.$$

It is called the *tensor product* of \mathcal{A}_1 and \mathcal{A}_2 .

2.2. Generalized wreath product. Let $S = U/L$ be a section of an S-ring \mathcal{A} . We say that \mathcal{A} is an *S-wreath product* if the group L is normal in G and $L \leq \text{rad}(X)$ for all basic sets X outside U ; in this case we write

$$(7) \quad \mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L},$$

and omit S when $|S| = 1$; in the latter case \mathcal{A} is called *wreath product*. When an explicit reference to the section S is not important, we use the term *generalized wreath product*. The *S-wreath product* is *nontrivial* or *proper* if $1 \neq L$ and $U \neq G$.

Theorem 2.1. *Let $S = U/L$ be a section of a group G , and let \mathcal{A}_1 and \mathcal{A}_2 be S-rings over the groups U and G/L respectively such that S is both an \mathcal{A}_1 - and an \mathcal{A}_2 -section with*

$$(\mathcal{A}_1)_S = (\mathcal{A}_2)_S.$$

Then the set of S-rings \mathcal{A} such that $\mathcal{A}_U = \mathcal{A}_1$ and $\mathcal{A}_{G/L} = \mathcal{A}_2$ has the smallest element. Moreover, it is a unique S-wreath product in this set.

Proof. It was proved in [3, Theorem 3.1] that the *S-wreath product* \mathcal{A} from (7) is uniquely determined and belongs to the set of S-rings from the theorem statement. For any other S-ring \mathcal{A}' from this set the preimage in $\mathcal{S}(\mathcal{A}')$ of at least one basic set in $\mathcal{S}(\mathcal{A}_2)$ outside U/L contains at least two basic sets, whereas the same preimage in $\mathcal{S}(\mathcal{A})$ consists of one element. This proves that $\mathcal{A}' \geq \mathcal{A}$, and hence the minimality of \mathcal{A} . ■

2.3. Duality. Let \mathcal{A} be an S-ring over a finite abelian group G and \widehat{G} the group dual to G , i.e. the group of all irreducible complex characters of G . Given $S \subset G$ and $\chi \in \widehat{G}$ set

$$(8) \quad \chi(S) = \sum_{s \in S} \chi(s).$$

Characters $\chi_1, \chi_2 \in \widehat{G}$ are called equivalent if $\chi_1(S) = \chi_2(S)$ for all $S \in \mathcal{S}(\mathcal{A})$. Denote by $\widehat{\mathcal{S}}$ the set of classes of this equivalence relation. Then the submodule of $\mathbb{Z}\widehat{G}$ spanned by the elements $\xi(X)$, $X \in \widehat{\mathcal{S}}$, is an S-ring over \widehat{G} (see [1, Theorem 6.3]). This ring is called *dual* to \mathcal{A} and is denoted by $\widehat{\mathcal{A}}$. Obviously, $\mathcal{S}(\widehat{\mathcal{A}}) = \widehat{\mathcal{S}}$. Moreover, $\text{rk}(\widehat{\mathcal{A}}) = \text{rk}(\mathcal{A})$ and

$$(9) \quad \mathcal{G}(\widehat{\mathcal{A}}) = \{H^\perp : H \in \mathcal{G}(\mathcal{A})\}$$

where $H^\perp = \{\chi \in \widehat{G} : H \leq \ker(\chi)\}$. It is also true that the S-ring dual to $\widehat{\mathcal{A}}$ is equal to \mathcal{A} .

It is easily seen that the mapping from $\text{Aut}(G)$ to $\text{Aut}(\widehat{G})$ that takes σ to $\widehat{\sigma}$ defined by $\chi^{\widehat{\sigma}}(g) = \chi(g^\sigma)$, is a group isomorphism. The image of a group $K \leq \text{Aut}(G)$ is denoted by \widehat{K} .

Theorem 2.2. *Let $\mathcal{A} = \text{Cyc}(K, G)$ where $K \leq \text{Aut}(G)$. Then $\widehat{\mathcal{A}} = \text{Cyc}(\widehat{K}, \widehat{G})$.*

Proof. Let $X \in \text{Orb}(\widehat{K}, \widehat{G})$. Then given $\chi_1, \chi_2 \in X$ there exists $\sigma \in K$ such that $\chi_1 = \chi_2^{\widehat{\sigma}}$. Since $S = S^\sigma$ for each basic set S of \mathcal{A} , this implies that

$$\chi_1(S) = \chi_2^{\widehat{\sigma}}(S) = \chi_2(S^\sigma) = \chi_2(S), \quad S \in \mathcal{S}(\mathcal{A}).$$

Therefore $\text{Cyc}(\widehat{K}, \widehat{G}) \geq \widehat{\mathcal{A}}$. Replacing here K and G by \widehat{K} and \widehat{G} respectively, we conclude that the S-ring \mathcal{A} contains the S-ring dual to $\text{Cyc}(\widehat{K}, \widehat{G})$. By duality this implies that $\widehat{\mathcal{A}} \geq \text{Cyc}(\widehat{K}, \widehat{G})$. Thus $\widehat{\mathcal{A}} = \text{Cyc}(\widehat{K}, \widehat{G})$. ■

Some more properties of the dual S-ring are contained in the following statement proved in [11, Theorems 2.4, 2.5]. In what follows given a section $S = U/L$ the group \widehat{S} is canonically identified with the section L^\perp/U^\perp of the group \widehat{G} that is called the section *dual* to S . In particular, if $G = G_1 \times G_2$, then $\widehat{G} = \widehat{G}_1 \times \widehat{G}_2$.

Theorem 2.3. *Let \mathcal{A} be an S-ring over an abelian group G . Then*

- (1) $\widehat{\mathcal{A}}_S = \widehat{\mathcal{A}}_{\widehat{S}}$ for any $S \in \mathfrak{S}(\mathcal{A})$,
- (2) $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ if and only if $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_1 \otimes \widehat{\mathcal{A}}_2$,
- (3) \mathcal{A} is an S-wreath product if and only if $\widehat{\mathcal{A}}$ is an \widehat{S} -wreath product. ■

3. SIMILARITIES, ISOMORPHISMS AND SCHURITY

In this section we follow papers [3, 4, 9] except for terminology: similarities and isomorphisms defined below were called in [3, 4] weak and strong isomorphisms respectively (see also the remark in Subsection 3.2). The reader familiar with association scheme theory will see that the definitions of similarity, isomorphism, etc. given in this section, are compatible with those used for Cayley schemes (see [9]).

3.1. Similarities. Let \mathcal{A} and \mathcal{A}' be S-rings over groups G and G' respectively. A ring isomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ is called *similarity* from \mathcal{A} to \mathcal{A}' , if for any $X \in \mathcal{S}(\mathcal{A})$ there exists $X' \in \mathcal{S}(\mathcal{A}')$ such that

$$\varphi\left(\sum_{x \in X} x\right) = \sum_{x' \in X'} x'.$$

It follows from the definition that the mapping $X \mapsto X'$ is a bijection from $\mathcal{S}(\mathcal{A})$ onto $\mathcal{S}(\mathcal{A}')$. This bijection is naturally extended to a bijection between \mathcal{A} - and \mathcal{A}' -sets, that takes $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{A}')$, and hence $\mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}(\mathcal{A}')$. The images of an \mathcal{A} -set X and \mathcal{A} -section S are denoted by X^φ and S^φ respectively. For any such S the similarity φ induces a similarity

$$\varphi_S : \mathcal{A}_S \rightarrow \mathcal{A}'_{S'} \quad \text{where} \quad S' = S^\varphi.$$

The set of all similarities from \mathcal{A} to \mathcal{A}' is denoted by $\Phi(\mathcal{A}, \mathcal{A}')$; we also set $\Phi(\mathcal{A}) = \Phi(\mathcal{A}, \mathcal{A})$.

The above bijection between the \mathcal{A} - and \mathcal{A}' -sets is in fact an isomorphism of the corresponding lattices. It follows that given an \mathcal{A} -set X we have

$$(10) \quad \langle X^\varphi \rangle = \langle X \rangle^\varphi \quad \text{and} \quad \text{rad}(X^\varphi) = \text{rad}(X)^\varphi.$$

These equalities together with the obvious equation $X = X \text{rad}(X)$, immediately imply the following statement.

Lemma 3.1. *Any similarity of an S-ring is uniquely determined by its restrictions to principal sections.■*

Any automorphism $\sigma \in \text{Aut}(G)$ can be extended linearly to a ring automorphism φ_σ of $\mathbb{Z}G$. Thus the lemma below immediately follows from the definition of similarity.

Lemma 3.2. *The mapping $\sigma \mapsto \varphi_\sigma$ is a group isomorphism from $\text{Aut}(G)$ onto $\Phi(\mathbb{Z}G)$.■*

Let Φ be a group of similarities of the S-ring \mathcal{A} . Then the set \mathcal{A}^Φ of the elements in \mathcal{A} left fixed under the action of Φ , is obviously an S-ring over G for which

$$\mathcal{S}(\mathcal{A}^\Phi) = \{X^\Phi : X \in \mathcal{S}(\mathcal{A})\}$$

where $X^\Phi = \bigcup_{\varphi \in \Phi} X^\varphi$. When $\mathcal{A} = \mathbb{Z}G$, from Lemma 3.2 it follows that \mathcal{A}^Φ is a cyclotomic S-ring.

We complete the subsection by describing the similarities of generalized wreath products; mostly this was done in [3].

Theorem 3.3. *Let \mathcal{A} and \mathcal{A}' be S-rings over abelian groups G and G' . Suppose that \mathcal{A} is an S-wreath product where $S = U/L$. Then*

- (1) *for any similarity $\varphi \in \Phi(\mathcal{A}, \mathcal{A}')$ the S-ring \mathcal{A}' is the S' -wreath product where $S' = S^\varphi$,*
- (2) *if \mathcal{A}' is an S' -wreath product, then the mapping $\varphi \mapsto (\varphi_U, \varphi_{G/L})$ induces a bijection from the set $\{\varphi \in \Phi(\mathcal{A}, \mathcal{A}') : S' = S^\varphi\}$ to the set*

$$\{(\varphi_1, \varphi_2) \in \Phi(\mathcal{A}_U, \mathcal{A}'_{U'}) \times \Phi(\mathcal{A}_{G/L}, \mathcal{A}'_{G'/L'}) : (\varphi_1)_S = (\varphi_2)_S\}.$$

where $U' = U^\varphi$ and $L' = L^\varphi$.

Proof. Statement (2) follows from [3, Theorem 3.3]. To prove statement (1) it suffices to verify that $L'X' = X'$ for all $X' \in \mathcal{S}(\mathcal{A}')_{G' \setminus U'}$. However, this is true because $(G \setminus U)^\varphi = G' \setminus U'$ and $LX = X$ for all $X \in \mathcal{S}(\mathcal{A})_{G \setminus U}$.■

3.2. Isomorphisms. Let \mathcal{A} and \mathcal{A}' be S-rings over groups G and G' respectively. A bijection $f : G \rightarrow G'$ is called an *isomorphism* from \mathcal{A} onto \mathcal{A}' if there exists a similarity $\varphi \in \Phi(\mathcal{A}, \mathcal{A}')$ such that given $X \in \mathcal{S}(\mathcal{A})$ we have

$$(11) \quad f(Xy) = X^\varphi f(y) \quad \text{for all } y \in G,$$

or, equivalently, $f(x)f(y)^{-1} \in X^\varphi$ for all $x, y \in G$ with $xy^{-1} \in X$. In this case we also say that f *induces* φ . Clearly, any isomorphism induces a uniquely determined similarity. The set of all isomorphisms and isomorphisms with a fixed φ are denoted by $\text{Iso}(\mathcal{A}, \mathcal{A}')$ and $\text{Iso}(\mathcal{A}, \mathcal{A}', \varphi)$ respectively.

It follows from the definition that the isomorphism f that takes 1_G to $1_{G'}$, takes $\mathcal{S}(\mathcal{A})$ to $\mathcal{S}(\mathcal{A}')$ and satisfies the condition $f(Xy) = f(X)f(y)$ for all $X \in \mathcal{S}(\mathcal{A})$ and $y \in G$. Therefore f is a strong isomorphism from \mathcal{A} to \mathcal{A}' in the sense of [3].

Conversely, according to that paper any strong isomorphism f induces a similarity φ such that $f(X) = X^\varphi$, and hence satisfies (11). Thus S-rings are isomorphic if and only if they are strongly isomorphic (see equality (12) below).

It immediately follows from the definition that

$$(12) \quad G_{right} \text{Iso}(\mathcal{A}, \mathcal{A}', \varphi) G'_{right} = \text{Iso}(\mathcal{A}, \mathcal{A}', \varphi).$$

In particular, not every isomorphism takes 1_G to $1_{G'}$. But even if it does, it is not necessarily a Cayley isomorphism. However, $|G| = |G'|$ because every similarity preserves the order of the underlying group. Moreover, since equality (11) obviously holds also for any \mathcal{A} -set X , and, in particular, every isomorphism preserves the right cosets of any \mathcal{A} -group.

Lemma 3.4. *In the above notation let H be an \mathcal{A} -group and $H' = H^\varphi$. Then*

$$hf^X h' \in \text{Iso}(\mathcal{A}_H, \mathcal{A}'_{H'}, \varphi_H)$$

for all $X \in G/H$, $h \in G_{H \rightarrow X}$ and $h' \in G'_{X' \rightarrow H'}$ where $X' = X^f$.

Proof. Denote by g and g' the permutations from G_{right} and G'_{right} such that $g^H = h$ and $(g')^{X'} = h'$, respectively. Then by (12) the bijection $gf g' : G \rightarrow G'$ induces the similarity φ . So the bijection $hf^X h' = (gf g')^H$ induces the similarity φ_H as required. ■

The following statement characterizes the isomorphisms of a generalized wreath product.

Theorem 3.5. *Let \mathcal{A} and \mathcal{A}' be S-rings over abelian groups G and G' and $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ a similarity. Suppose that \mathcal{A} is a U/L -wreath product. Then the set $\text{Iso}(\mathcal{A}, \mathcal{A}', \varphi)$ consists of all bijections $f : G \rightarrow G'$ such that $(G/U)^f = G'/U'$, $(G/L)^f = G'/L'$ where $U' = U^\varphi$ and $L' = L^\varphi$, and*

$$(13) \quad f^{G/L} \in \text{Iso}(\mathcal{A}_{G/L}, \mathcal{A}'_{G/L}, \varphi_{G/L}), \quad gf^X g' \in \text{Iso}(\mathcal{A}_U, \mathcal{A}'_U, \varphi_U)$$

for all $X \in G/U$ and some $g \in G_{U \rightarrow X}$ and $g' \in G'_{X' \rightarrow U'}$ where $X' = X^f$.

Proof. Set $F = \text{Iso}(\mathcal{A}, \mathcal{A}', \varphi)$ and denote by F' the set of all bijections $f : G \rightarrow G'$ satisfying (13). Then the inclusion $F \supset F'$ immediately follows from the basic properties of similarities and Lemma 3.4. Conversely, let $f \in F'$. We have to verify that equality (11) holds for all $X \in \mathcal{S}(\mathcal{A})$ and $y \in G$. Suppose first that $X \subset G \setminus U$. Then from the equality $(G/L)^f = G'/L'$ and the first relation in (13) it follows that

$$f^{G/L}(X^\pi y^\pi) = (X^\pi)^{\varphi_{G/L}} f^{G/L}(y^\pi)$$

where $\pi = \pi_{G/L}$ and $y \in G$. After taking the preimages of both sides in the latter equality we obtain that $f(XyL) = (XL)^\varphi f(yL)$. On the other hand, $L \leq \text{rad}(X)$. Due to (10) this implies that $L' \leq \text{rad}(X^\varphi)$. Thus

$$f(Xy) = f(XyL) = (XL)^\varphi f(yL) = X^\varphi f(y)L' = X^\varphi f(y),$$

which proves the required statement in our case. Let now $X \subset U$ and $y \in G$. Then from the equality $(G/U)^f = G'/U'$ and the second relation in (13) it follows that

$$X^{gf^Y g'} = X^\varphi$$

for some $g \in G_{U \rightarrow Y}$ and $g' \in G'_{Y' \rightarrow U'}$ where $Y = Uy$ and $Y' = Y^f$. By Lemma 3.4 without loss of generality we can assume that the permutations g and g' are induced

by multiplications by y and $(y^f)^{-1}$ respectively. Then $X^{gf^Y g'} = (Xy)^f (y^f)^{-1}$, and hence

$$f(Xy) = (Xy)^f = X^\varphi y^f = X^\varphi f(y)$$

and we are done. ■

Let \mathcal{K} be a class of S-rings closed under Cayley isomorphisms. Following [7] an S-ring \mathcal{A} is called separable with respect to \mathcal{K} if $\text{Iso}(\mathcal{A}, \mathcal{A}', \varphi) \neq \emptyset$ for all similarities $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ where $\mathcal{A}' \in \mathcal{K}$. In this paper we say that a circulant S-ring is *separable* if it is separable with respect to the class of all circulant S-rings.

3.3. Schurity. Let G be a finite group. It was proved by Schur (see [17, Theorem 24.1]) that any group $\Gamma \leq \text{Sym}(G)$ that contains G_{right} produces an S-ring \mathcal{A} over G such that

$$\mathcal{S}(\mathcal{A}) = \text{Orb}(\Gamma_1, G)$$

where $\Gamma_1 = \{\gamma \in \Gamma : 1^\gamma = 1\}$ is the stabilizer of the point $1 = 1_G$ in Γ . Any such S-ring is called *schurian*. Group rings and S-rings of rank 2 are obviously schurian.

Let Γ and Δ be permutation groups on G such that $G_{\text{right}} \leq \Gamma \cap \Delta$. Then it is easily seen that the S-ring associated with the group $\langle \Gamma, \Delta \rangle$ equals the intersection of S-rings associated with Γ and Δ . It follows that the intersection of schurian S-rings is schurian. Therefore so is the S-ring

$$(14) \quad \text{Sch}(\mathcal{A}) = \bigcap_{\mathcal{A}' \geq \mathcal{A}, \mathcal{A}' \text{ is schurian}} \mathcal{A}'.$$

It is called the *schurian closure* of \mathcal{A} . Clearly, $\text{Sch}(\mathcal{A}) \geq \mathcal{A}$, and the equality is attained if and only if the S-ring \mathcal{A} is schurian.

The schurity concept is closely related to automorphisms of an S-ring. In contrast to a common algebraic tradition the automorphism group of an S-ring \mathcal{A} is not defined to be $\text{Iso}(\mathcal{A}, \mathcal{A})$; in accordance with a combinatorial tradition we set $\text{Aut}(\mathcal{A}) = \text{Iso}(\mathcal{A}, \mathcal{A}, \text{id}_{\mathcal{A}})$. Thus $f \in \text{Aut}(\mathcal{A})$ if and only if

$$(15) \quad f(Xy) = Xf(y), \quad X \in \mathcal{S}(\mathcal{A}), \quad y \in G.$$

The latter is equivalent to say that given $X \in \mathcal{S}(\mathcal{A})$ we have $f(x)f(y)^{-1} \in X$ whenever $xy^{-1} \in X$. Therefore $\text{Aut}(\mathcal{A})$ can also be defined as the automorphism group of the colored Cayley graph corresponding to the partition $\mathcal{S}(\mathcal{A})$ of the group G (cf. Introduction).

It follows that any basic set of \mathcal{A} is invariant with respect to the group $\text{Aut}(\mathcal{A})_1$, whereas equality (12) shows that $G_{\text{right}} \leq \text{Aut}(\mathcal{A})$. Moreover, the group $\text{Aut}(\mathcal{A})$ is the largest subgroup of $\text{Sym}(G)$ that satisfies these two properties. Now, let Γ be a permutation group the schurian S-ring $\text{Sch}(\mathcal{A})$ is associated with. Then since $\text{Sch}(\mathcal{A}) \geq \mathcal{A}$, the maximality of $\text{Aut}(\mathcal{A})$ implies that

$$\Gamma \leq \text{Aut}(\mathcal{A}).$$

It follows that $\text{Sch}(\mathcal{A})$ contains the S-ring associated with $\text{Aut}(\mathcal{A})$. Therefore these S-rings are equal (see (14)). In fact, this shows that the closure of the S-ring \mathcal{A} with respect to Galois correspondence (1) coincides with $\text{Sch}(\mathcal{A})$. Thus the above definition of a schurian S-ring is compatible with that given in Introduction.

Let $f \in \text{Aut}(\mathcal{A})_1$. Then any \mathcal{A} -set (in particular, \mathcal{A} -group) is invariant with respect to the automorphism f . Moreover, for any \mathcal{A} -section S we have $f^S \in \text{Aut}(\mathcal{A}_S)$. In particular, the S-ring \mathcal{A}_S is schurian whenever so is \mathcal{A} .

The following result proved in [9, Corollary 5.7] gives a criterion for the schurity of generalized wreath products that will be repeatedly used throughout the paper. Below we set

$$\mathcal{M}(\mathcal{A}) = \{\Gamma \leq \text{Sym}(G) : \Gamma \approx_2 \text{Aut}(\mathcal{A}) \text{ and } G_{\text{right}} \leq \Gamma\}.$$

Theorem 3.6. *Let \mathcal{A} be an S-ring over an abelian group G . Suppose that \mathcal{A} is an S-wreath product where $S = U/L$. Then \mathcal{A} is schurian if and only if so are the S-rings $\mathcal{A}_{G/L}$ and \mathcal{A}_U and there exist groups $\Delta_0 \in \mathcal{M}(\mathcal{A}_{G/L})$ and $\Delta_1 \in \mathcal{M}(\mathcal{A}_U)$ such that $(\Delta_0)^S = (\Delta_1)^S$.*

It should be remarked that a permutation $f \in \text{Sym}(G)$ preserving every basic set of \mathcal{A} does not necessarily belong to $\text{Aut}(\mathcal{A})$. However, as the lemma below shows, this is so when, for example, $f \in \text{Aut}(G)$.

Lemma 3.7. *Let \mathcal{A} be an S-ring and $\mathcal{A}' \geq \mathcal{A}$. Then the group of all isomorphisms of \mathcal{A}' that fix every element of \mathcal{A} , is a subgroup of $\text{Aut}(\mathcal{A})$.*

Proof. Let f be an isomorphism of \mathcal{A}' and φ the similarity of \mathcal{A}' induced by f . Then $f(X'y) = (X')^\varphi f(y)$ for all \mathcal{A}' -sets X' and $y \in G$. Since $\mathcal{A}' \geq \mathcal{A}$, this is true for all $X' \in \mathcal{S}(\mathcal{A})$. So if f fixes every element of \mathcal{A} , then φ is identical on \mathcal{A} , and hence $f(Xy) = Xf(y)$ for all $X \in \mathcal{S}(\mathcal{A})$. Thus $f \in \text{Aut}(\mathcal{A})$. ■

4. SECTIONS IN S-RINGS

4.1. Projective equivalence. Let G be a group. Denote by $\mathcal{G}(G)$ the set of its subgroups, and by $\mathfrak{S}(G)$ the set of its sections, i.e. quotients of subgroups of G . When this does not lead to misunderstanding, we write H instead of $H/1$ where $H \in \mathcal{G}(G)$, and identify $\mathfrak{S}(S)$ with the corresponding subset of $\mathfrak{S}(G)$ where $S \in \mathfrak{S}(G)$.

A section U'/L' is called a *subsection* of a section U/L if $U' \leq U$ and $L' \geq L$; in this case we write $U'/L' \preceq U/L$. This defines a partial order on the set $\mathfrak{S}(G)$. This order has the greatest element $G/1$; any minimal element is of the form H/H where $H \in \mathcal{G}(G)$.

A section U/L is called a *multiple* of a section U'/L' if

$$(16) \quad L' = U' \cap L \quad \text{and} \quad U = U'L.$$

The *projective equivalence* relation " \sim " on the set $\mathfrak{S}(G)$ is defined to be the transitive closure of the relation "to be a multiple". Any two projectively equivalent sections are obviously isomorphic as groups. The set of all equivalence classes is denoted by $\mathcal{P}(G)$. Under a *quasisubsection* of a section S we mean any section which is projectively equivalent to a subsection of S .

Given a section $S = H/K$ of a group G we define a surjection $\rho_{G,S}$ from the subgroups of G to the subgroups of S by

$$\rho_{G,S}(U) = (U \cap H)K/K.$$

Let us extend this mapping to $\mathfrak{S}(G)$ by $\rho_{G,S}(U/L) = \rho_{G,S}(U)/\rho_{G,S}(L)$. Using the above identification of $\mathfrak{S}(S)$ with the corresponding subset of $\mathfrak{S}(G)$, we obtain that

$$(17) \quad \rho_{G,S}(U/L) = (U \cap H)K / (L \cap H)K.$$

Clearly, this mapping is identical on the set $\mathfrak{S}(S)$. Moreover, $\rho_{G,S}$ induces a surjective homomorphism of the corresponding partially ordered sets. To simplify notations we will write T_S instead of $\rho_{G,S}(T)$ when the group G is fixed.

If the group G is abelian, then the set $\mathcal{G}(G)$ of all subgroups of G forms a modular lattice⁴ in which the join and meet of H and K are defined as HK and $H \cap K$ respectively [16].

Lemma 4.1. *Let G be an abelian group. Then given $S, T \in \mathfrak{S}(G)$ we have $S_T \sim T_S$.*

Proof. Let $S = H/K$ and $T = U/L$. Then by the definition of S_T and T_S we have

$$S_T = (H \cap U)L / (K \cap U)L \quad \text{and} \quad T_S = (U \cap H)K / (L \cap H)K.$$

A straightforward check shows that both S_T and T_S are multiples of the section $(H \cap U)/(H \cap U \cap KL)$ (we made use the fact the lattice $\mathcal{G}(G)$ is modular). Therefore the sections S_T and T_S are projectively equivalent. ■

4.2. Restrictions. Let \mathcal{A} be an S-ring over a group G . Set

$$\mathcal{P}(\mathcal{A}) = \{C \cap \mathfrak{S}(\mathcal{A}) : C \in \mathcal{P}(G), C \cap \mathfrak{S}(\mathcal{A}) \neq \emptyset\}.$$

Then $\mathcal{P}(\mathcal{A})$ forms a partition of the set $\mathfrak{S}(\mathcal{A})$ into classes of projectively equivalent \mathcal{A} -sections. It should be noted that if $\mathcal{A}' \geq \mathcal{A}$, then $\mathfrak{S}(\mathcal{A}') \supset \mathfrak{S}(\mathcal{A})$ and each class of projectively equivalent \mathcal{A} -sections is contained in a unique class of projectively equivalent \mathcal{A}' -sections.

Theorem 4.2. *Let \mathcal{A} be an S-ring over a group G . Then given projectively equivalent \mathcal{A} -sections S and T there exists a Cayley isomorphism f from \mathcal{A}_S onto \mathcal{A}_T such that $(\gamma^S)^f = \gamma^T$ for all $\gamma \in \text{Aut}(\mathcal{A})$ leaving the point 1_G fixed.*

Proof. Follows from [9, Lemma 3.1, Theorem 3.2]. ■

Obviously, any two sections of a class $C \in \mathcal{P}(G)$ have the same order; we call it the *order* of this class. If, in addition, $C \in \mathcal{P}(\mathcal{A})$ where \mathcal{A} is an S-ring over G , then from Theorem 4.2 it follows that all sections in C have the same rank r , and also if C contains a primitive (resp. cyclotomic, dense) section, then all sections in C are primitive (resp. cyclotomic, dense). In these cases we say that C is a class of rank r , and a primitive (resp. cyclotomic, dense) class.

In general, an \mathcal{A} -section projectively equivalent to a principal \mathcal{A} -section is not principal. However, at least in the cyclic group case such a section is *subprincipal*, i.e. a subsection of a principal \mathcal{A} -section (Lemma 5.3). But even in this case the class of subprincipal sections is not closed under the projective equivalence. To avoid this inconvenience we define an \mathcal{A} -section to be *quasiprincipal* (resp. *quasisubprincipal*) if it is projectively equivalent to a principal (resp. subprincipal) \mathcal{A} -section.

Let $S \in \mathfrak{S}(\mathcal{A})$. Then the mapping $\rho_{G,S} : T \mapsto T_S$ defined in Subsection 4.1 induces a mapping from $\mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}(\mathcal{A}_S)$ that is denoted by the same letter. The following statement shows that it preserves generalized wreath products.

⁴We recall that a lattice is modular, if $x \vee (y \wedge z) = (x \vee y) \wedge z$ whenever $x \leq z$.

Theorem 4.3. *Let \mathcal{A} be an S -ring over an abelian group G . Suppose that \mathcal{A} is the T -wreath product where $T \in \mathfrak{S}(\mathcal{A})$. Then \mathcal{A}_S is the T_S -wreath product for all $S \in \mathfrak{S}(\mathcal{A})$.*

Proof. Let $T = U/L$ and $S = H/K$. We have to prove that $L_S \leq \text{rad}(Y)$ for all $Y \in \mathcal{A}_{S \setminus U_S}$. However, any $Y \in \mathcal{S}(\mathcal{A}_S)$ is of the form XK/K for some $X \in \mathcal{S}(\mathcal{A}_H)$. If, in addition, $Y \in \mathcal{A}_{S \setminus U_S}$, then $X \in \mathcal{A}_{G \setminus U}$, and hence $L \leq \text{rad}(X)$ because \mathcal{A} is the T -wreath product. Since $\text{rad}(X) \leq \langle X \rangle \leq H$, this implies that

$$L_S \leq \text{rad}(X)_S = \text{rad}(X)K/K \leq \text{rad}(XK/K) = \text{rad}(Y)$$

as required. ■

4.3. Duality. Let G be an abelian group. Then the mapping $H \mapsto H^\perp$ induces a lattice antiisomorphism from $\mathcal{G}(G)$ onto $\mathcal{G}(\widehat{G})$ (see e.g. [16]).

Lemma 4.4. *For any sections $S, T \in \mathfrak{S}(G)$ the following statements hold:*

- (1) $S \sim T$ if and only if $\widehat{S} \sim \widehat{T}$; moreover, T is a multiple of S if and only if \widehat{S} is a multiple of \widehat{T} ,
- (2) $\widehat{\rho_{G,S}(T)} = \rho_{\widehat{G},\widehat{S}}(\widehat{T})$,
- (3) $S \preceq T$ if and only if $\widehat{S} \preceq \widehat{T}$.

Proof. Let $S = H/K$ and $T = U/L$. To prove statement (1) without loss of generality we can assume that T is a multiple of S . Then from (16) it follows that

$$K^\perp = H^\perp L^\perp \quad \text{and} \quad U^\perp = H^\perp \cap L^\perp,$$

which means that $\widehat{S} = K^\perp/H^\perp$ is a multiple of $\widehat{T} = L^\perp/U^\perp$ as required. Statement (2) follows from (17) and the modularity of the lattice $\mathcal{G}(G)$:

$$\begin{aligned} \widehat{\rho_{G,S}(T)} &= ((U \cap H)K)^\perp / ((L \cap H)K)^\perp = ((UK) \cap H)^\perp / ((LK) \cap H)^\perp = \\ &= (U^\perp \cap K^\perp)H^\perp / (L^\perp \cap K^\perp)H^\perp = \rho_{\widehat{G},\widehat{S}}(\widehat{T}). \end{aligned}$$

Statement (3) is obvious. ■

Let \mathcal{A} be an S -ring over G . Given a class $C \in \mathcal{P}(\mathcal{A})$ we define the dual class by

$$\widehat{C} = \{\widehat{S} : S \in C\}.$$

Then from statement (1) of Lemma 4.4 it follows that $\widehat{C} \in \mathcal{P}(\widehat{\mathcal{A}})$. Moreover, the classes \widehat{C} and C have the same order and rank, and if one of them is primitive (resp. cyclotomic, dense), then so is the other one (see equality (9) and Theorem 2.2).

5. CIRCULANT S -RINGS

5.1. General theory. We begin with a well-known result on circulant primitive S -rings, that goes back to Burnside and Schur. Despite the fact that they dealt with groups, their results can be interpreted as results on schurian circulant S -rings. Moreover, the Schur method works in the non-schurian case as well (see, e.g. [6]).

Theorem 5.1. *Any circulant primitive S -ring is of rank 2, or a cyclotomic S -ring over a group of prime order.*

Let \mathcal{A} be an S-ring over a cyclic group G . Then

$$X^\sigma \in \mathcal{S}(\mathcal{A}) \quad \text{for all } X \in \mathcal{S}(\mathcal{A}), \sigma \in \text{Aut}(G),$$

see [17, Theorem 23.9]. Let now X be a *highest* basic set of \mathcal{A} , i.e. one containing a generator of G . Then the above statement implies that the group $\text{rad}(X)$ does not depend on the choice of X . It is called the *radical* of \mathcal{A} and denoted by $\text{rad}(\mathcal{A})$. The following statement proved by Leung and Man is a cornerstone of the circulant S-ring theory (see [4, Corollaries 5.5, 6.4]).

Theorem 5.2. *Let \mathcal{A} be a circulant S-ring. Then the following two statements hold:*

- (1) $\text{rad}(\mathcal{A}) \neq 1$ if and only if \mathcal{A} is a proper generalized wreath product,
- (2) $\text{rad}(\mathcal{A}) = 1$ if and only if \mathcal{A} is the tensor product of a cyclotomic S-ring with trivial radical and S-rings of rank 2. ■

Any principal \mathcal{A} -section S is obviously a section with trivial radical in the sense that $\text{rad}(\mathcal{A}_S) = 1$. The converse is not true, but we have the following statement.

Lemma 5.3. *In a circulant S-ring, any section with trivial radical is subprincipal.*

Proof. Suppose that $\text{rad}(\mathcal{A}_S) = 1$ where $S = U/L$ is a section of a circulant S-ring \mathcal{A} . We observe that every highest basic set X of the S-ring \mathcal{A}_U produces a highest basic set $\pi(X)$ of the S-ring \mathcal{A}_S where $\pi = \pi_S$. Thus

$$\pi(\text{rad}(X)) \leq \text{rad}(\pi(X)) = \text{rad}(\mathcal{A}_S) = 1.$$

It follows that $\text{rad}(X) \leq L$. Therefore $S \preceq T$ where $T = U/\text{rad}(X)$. Since $U = \langle X \rangle$, the \mathcal{A} -section T is principal, and we are done. ■

From Theorem 4.2 it follows that the set of all \mathcal{A} -sections with trivial radical is closed with respect to the projective equivalence. Another property of the projective equivalence for circulant S-rings is given in the following statement proved in [5, Lemma 5.2]. Below a section of a class $C \in \mathcal{P}(\mathcal{A})$ is called the *smallest* (respectively, *largest*) one if every section of C , is a multiple of it (respectively, if it is a multiple of every section of C).

Theorem 5.4. *Any class of projectively equivalent sections of a circulant S-ring has the largest and smallest elements. ■*

5.2. Quasidense S-rings. A circulant S-ring \mathcal{A} is called *quasidense*, if any primitive \mathcal{A} -section is of prime order. Any dense S-ring is obviously quasidense. Moreover, in the quasidense case any minimal \mathcal{A} -group is of prime order, any maximal \mathcal{A} -group is of prime index, and the S-ring \mathcal{A}_S is dense for any \mathcal{A} -section S of prime power order.

Theorem 5.5. *A circulant S-ring is quasidense if and only if it contains no rank 2 sections of composite order.*

Proof. By Theorem 5.1 a section of composite order is of rank 2 if and only if it is primitive. Thus the required statement immediately follows from the definition of a quasidense S-ring. ■

It is easily seen that the class of quasidense S-rings is closed under restriction to a section, and the tensor and generalized wreath products.

Theorem 5.6. *Any extension of quasidense circulant S-ring is quasidense.*

Proof. Let \mathcal{A}' be a non-quasidense extension of a quasidense S-ring \mathcal{A} over a cyclic group G . Then by Theorem 5.5 the S-ring \mathcal{A}' contains a rank 2 section $S = U/L$ of composite order. Denote by H an \mathcal{A} -group of prime order (such a group does exist because \mathcal{A} is quasidense). We claim that

$$(18) \quad H \cap U = 1 \quad \text{or} \quad H \leq L.$$

Indeed, if (18) is not true, then $H \leq U$ and $H \cap L = 1$ because the order of H is prime. It follows that the S-ring \mathcal{A}'_S contains the group HL/L of prime order $|H|$. The latter group does not equal S because the order of S is composite. However this is impossible because $\text{rk}(\mathcal{A}'_S) = 2$.

By (18) the S-ring $\mathcal{A}'_{G/H}$ contains the section $S' = UH/LH$ which is projectively equivalent to the section U/L by the modularity of the lattice $\mathcal{G}(G)$. By Theorem 4.2 this implies that

$$\text{rk}(\mathcal{A}'_{S'}) = \text{rk}(\mathcal{A}'_S) = 2.$$

Thus, $\mathcal{A}'_{G/H}$ is a non-quasidense extension of the quasidense S-ring $\mathcal{A}_{G/H}$ due to Theorem 5.5. Assuming without loss of generality that the order of the group G is minimal possible we come to a contradiction. ■

Theorem 5.7. *The class of quasidense circulant S-rings with trivial radical is closed with respect to taking extensions, and consists of cyclotomic, and hence dense S-rings.*

Proof. Let \mathcal{A} be a quasidense circulant S-ring with trivial radical. Then it is cyclotomic, and hence dense by [10, Theorem 3.1]. If now $\mathcal{A}' \geq \mathcal{A}$, then the S-ring \mathcal{A}' is dense, and hence quasidense. Suppose on the contrary that $\text{rad}(\mathcal{A}') \neq 1$. Then by statement (1) of Theorem 5.2 the S-ring \mathcal{A}' is a proper S' -wreath product for some \mathcal{A}' -section S' . This section is also \mathcal{A} -section by the density of \mathcal{A} . Therefore \mathcal{A} is the S' -wreath product, which is impossible by statement (1) of Theorem 5.2 because $\text{rad}(\mathcal{A}) = 1$. ■

The following theorem deduced from [10, Theorem 3.5] shows that any schurian quasidense circulant S-ring can be obtained from an appropriate solvable permutation group that "locally" has a rather simple form.

Theorem 5.8. *Let \mathcal{A} be a schurian quasidense circulant S-ring. Then there exists a group $\Gamma \in \mathcal{M}(\mathcal{A})$ such that $\Gamma^S = \text{Hol}_{\mathcal{A}}(S)$ for any quasisubprinciple \mathcal{A} -section S where $\text{Hol}_{\mathcal{A}}(S) = \text{Hol}(S) \cap \text{Aut}(\mathcal{A}_S)$.*

Proof. By [10, Theorem 3.5] there exists a group $\Gamma \in \mathcal{M}(\mathcal{A})$ such that $\Gamma^S = \text{Hol}_{\mathcal{A}}(S)$ for any \mathcal{A} -section S such that $\text{rad}(\mathcal{A}_S) = 1$. Since obviously $\text{Hol}_{\mathcal{A}}(S)^T = \text{Hol}_{\mathcal{A}}(T)$ for all \mathcal{A} -sections $T \preceq S$, the required statement follows from Theorem 4.2 for $\gamma \in \Gamma$. ■

5.3. Duality. The following two theorems establishing selfdual properties of a circulant S-ring will be used in Section 13 to prove Theorem 1.4.

Theorem 5.9. *Let \mathcal{A} be a circulant S-ring. Then*

- (1) $\text{rad}(\mathcal{A}) = 1$ if and only if $\text{rad}(\widehat{\mathcal{A}}) = 1$,
- (2) \mathcal{A} is quasidense if and only if $\widehat{\mathcal{A}}$ is quasidense.

Proof. Statement (1) immediately follows from statement (3) of Theorem 2.3 and Theorem 5.2. Statement (2) holds because the primitivity and the order of an \mathcal{A} -section are preserved under duality: the former by equality (9) and statement (1) of Theorem 2.3, whereas the latter by the definition of the dual section. ■

Theorem 5.10. *A section of a circulant S-ring is quasisubprincipal if and only if so is its dual.*

Proof. Let S be a quasisubprincipal section of a circulant S-ring \mathcal{A} . Then S is projectively equivalent to a subsection S' of a principal \mathcal{A} -section T . Without loss of generality we can assume that T is a \preceq -maximal principal \mathcal{A} -section. Then \widehat{T} is a \preceq -maximal $\widehat{\mathcal{A}}$ -section by statement (1) of Lemma 4.4. Moreover, by statement (2) of Theorem 5.9 we have $\text{rad}(\widehat{\mathcal{A}}_{\widehat{T}}) = 1$. Thus by Lemma 5.3 we conclude that \widehat{T} is a principal $\widehat{\mathcal{A}}$ -section. On the other hand, from statements (1) and (3) of Lemma 4.4 we have $\widehat{S} \sim \widehat{S}'$ and $\widehat{S}' \preceq \widehat{T}$. Thus \widehat{S} is a quasisubprincipal \mathcal{A} -section as required. The converse statement holds by duality. ■

6. LIFTING: NONDENSE CASE

We begin with a characterization of the dense circulant S-rings in terms of forbidden subsections. To do this we will say that an S-ring \mathcal{A} over a cyclic group G is *elementary nondense* if $|G|$ is a composite number and \mathcal{A} has rank 2, or $|G|$ is the product of two distinct primes and \mathcal{A} is a proper wreath product. In the former case the S-ring is not quasidense, whereas in the latter case it is. A section S of a circulant S-ring \mathcal{A} is called *elementary nondense* if the S-ring \mathcal{A}_S is elementary nondense.

Lemma 6.1. *The section that is dual or projectively equivalent to an elementary nondense section is elementary nondense.*

Proof. Follows from Theorems 2.3 and 4.2. ■

It is easily seen that a circulant S-ring that contains an elementary nondense section cannot be dense. This proves the “if” part of the following statement.

Theorem 6.2. *A circulant S-ring \mathcal{A} is not dense if and only if there exists an elementary nondense \mathcal{A} -section. In particular, any minimal nondense \mathcal{A} -section is elementary.*

Proof. To prove the “only if” part suppose that \mathcal{A} is not dense. If it is not quasidense, then it contains a rank 2 section S of composite order (Theorem 5.5). Since S is elementary nondense, we are done. Suppose that \mathcal{A} is quasidense and $S = V/K$ is a minimal nondense \mathcal{A} -section. Then there exists a non- \mathcal{A} -group $H \leq G$ such that

$$K < H < V.$$

The minimality of S implies that K is a maximal \mathcal{A} -group inside H . So after decreasing H (if necessary) without loss of generality we can assume that the number $p = |H/K|$ is prime. Next, by the quasidensity of $\mathcal{A}_{V/K}$ there exists an \mathcal{A} -group M such that $K \leq M \leq V$ and the number $q = |M/K|$ is prime. Since H is not an \mathcal{A} -group, it follows that $p \neq q$ and hence

$$H \cap M = K.$$

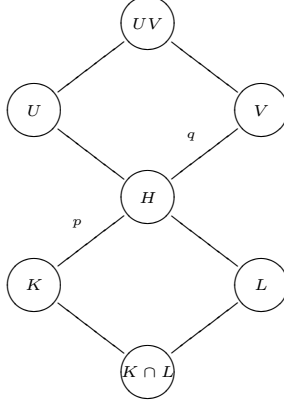


FIGURE 1.

Moreover, by the minimality of S , the section V/M is dense. Therefore MH is an \mathcal{A} -group. So $MH = V$ by the minimality of S . Thus $|V/K| = pq$. To complete the proof it suffices to note that a quasidense but not dense S-ring over a cyclic group of order pq has to be a proper wreath product (Theorem 5.2).■

In what follows \mathcal{A} is a quasidense S-ring over a cyclic group G .⁵ Suppose that $S_0 = V/K$ is an elementary nondense \mathcal{A} -section. Denote by H the unique \mathcal{A} -subgroup of G such that $K < H < V$. Then

$$\mathcal{A}_{S_0} = \mathcal{A}_{H/K} \wr \mathcal{A}_{V/H}.$$

The largest \mathcal{A} -section which is projectively equivalent to V/H , is obviously of the form UV/U for some \mathcal{A} -group $U = U(S_0)$; similarly, the smallest \mathcal{A} -section which is projectively equivalent to H/K , is of the form $L/(K \cap L)$ for some \mathcal{A} -group $L = L(S_0)$. (The existence of the largest and smallest sections follows from Theorem 5.4.) Clearly,

$$(19) \quad 1 < L \leq U < G.$$

The relevant part of the lattice of \mathcal{A} -groups is given in Fig. 1.

The definition of the section U/L associated with S_0 is uniform in the following sense. Let S be an \mathcal{A} -section that contains S_0 as a subsection. Since the mapping $\rho_{G,S}$ defined in Subsection 4.1 induces a lattice epimorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{A}_S)$, we have

$$(20) \quad \rho_{G,S}(U/L) = U_S/L_S$$

where U_S/L_S is the section of the S-ring \mathcal{A}_S defined by S_0 in the same way as the section U/L in the S-ring \mathcal{A} . In particular, $U_{S_0}/L_{S_0} = H/H$.

The following two statements will be used to prove the main result of this section (Theorem 6.5).

Lemma 6.3. *In the above notation the following statements hold:*

- (1) *the section S_0 is a quasisubsection of any \mathcal{A} -group $M \not\leq U$,*
- (2) *the section S_0 is a quasisubsection of any \mathcal{A} -section G/N with $N \not\geq L$.*

⁵Generalized wreath products arising in nonquasidense case had been studied in [9].

Proof. From Lemma 6.1 it follows that $\widehat{S_0}$ is an elementary nondense section of the S-ring $\widehat{\mathcal{A}}$. Therefore statement (2) follows from statement (1) by duality. To prove statement (1) we claim that UV/U is a \mathcal{A} -quasisubsection of M . Indeed, since $U \leq UM \cap UV \leq UV$ and the number $|UV/U| = |V/H|$ is prime, we have

$$UM \cap UV \in \{U, UV\}.$$

However, $UM \cap UV \neq U$, because otherwise UVM/UM is obviously a multiple of the section UV/U , which contradicts the maximality of it. Therefore $UM \cap UV = UV$, and hence $UV \leq UM$. But $U(M \cap UV) = UM \cap UV$ because the lattice $\mathcal{G}(\mathcal{A})$ is modular. Then

$$U(M \cap UV) = UM \cap UV = UV.$$

Since also obviously $U \cap M \cap UV = U \cap M$, we conclude that the section UV/U is a multiple of $(M \cap UV)/(M \cap U)$. This proves the claim because the latter is an \mathcal{A}_M -section.

Set V'/H' to be the smallest \mathcal{A} -section which is projectively equivalent to $V/H \sim UV/U$. Then by the claim in the previous paragraph, V'/H' is an \mathcal{A}_M -section. To complete the proof it suffices to verify that

$$V'/K' \sim V/K$$

where $K' = K \cap V'$. Let us prove that V/K is a multiple of V'/K' . Suppose on the contrary that $KV' \neq V$. Then $KV' \leq H$ because $K \leq KV' \leq V$ and H is the only \mathcal{A} -group strictly between K and V . Therefore $V' \leq H$. On the other hand, $V = V'H$ because V/H is a multiple of V'/H' . Thus $V = V'H \leq H$ which is impossible. ■

Corollary 6.4. *The section S_0 is a quasisubsection of any \mathcal{A} -section M/N with $M \not\leq U$ and $L \not\leq N$.*

Proof. By statement (1) of Lemma 6.3 the S-ring \mathcal{A}_M contains a section $S'_0 = V'/K'$ projectively equivalent to $S_0 = V/K$. Since the section S'_0 is elementary nondense (Lemma 6.1), there is a unique \mathcal{A} -group H' strictly between K' and V' . From the choice of the group L it follows that

$$H'/K' \sim H/K \sim L/K \cap L,$$

Moreover, $L \leq H' \leq M$. Since also $N \leq M$, the hypothesis of Lemma 6.3 is satisfied for $G = M$ and $S_0 = S'_0$. So by statement (2) of this lemma we conclude that the section M/N has a subsection projectively equivalent to $S'_0 \sim S_0$ as required. ■

From the Leung-Man theory it follows that any quasidense circulant S-ring that is not dense, is a proper generalized wreath product (see [4, Theorem 5.3]). The following theorem gives an explicit form of such a product.

Theorem 6.5. *Let \mathcal{A} be a quasidense circulant S-ring and S_0 an elementary nondense section. Then \mathcal{A} is a proper U/L -wreath product with $U = U(S_0)$ and $L = L(S_0)$. Moreover, any dense \mathcal{A} -section is either an \mathcal{A}_U -section or $\mathcal{A}_{G/L}$ -section.*

Proof. To prove the first statement it suffices to verify by (19) that

$$\text{rad}(X) \geq L \quad \text{for all } X \in \mathcal{S}(\mathcal{A})_{G \setminus U}.$$

Suppose on the contrary that this is not true for some X . Then the hypothesis of Corollary 6.4 holds for $M = \langle X \rangle$ and $N = \text{rad}(X)$. Therefore S_0 is a quasisubsection of M/N . Since S_0 is elementary nondense, this implies that the S-ring $\mathcal{A}_{M/N}$ is not dense. However, this is impossible because $\mathcal{A}_{M/N}$ is a quasidense S-ring with trivial radical (Theorem 5.7).

To prove the second statement suppose on the contrary that there exists a dense \mathcal{A} -section M/N which is neither an \mathcal{A}_U - nor $\mathcal{A}_{G/L}$ -section. Then S_0 is a quasisubsection of M/N by Corollary 6.4. Therefore the S-ring $\mathcal{A}_{M/N}$ is not dense. Contradiction. ■

The following auxiliary statement will be used in Section 10.

Lemma 6.6. *Let \mathcal{A}' be an extension of a quasidense circulant S-ring \mathcal{A} . Then $\mathcal{G}(\mathcal{A}') = \mathcal{G}(\mathcal{A})$ if and only if any elementary nondense section of \mathcal{A} is an elementary nondense section of \mathcal{A}' .*

Proof. The “only if” part is obvious. To prove the “if” part suppose on the contrary that there exists a group $H \in \mathcal{G}(\mathcal{A}') \setminus \mathcal{G}(\mathcal{A})$. Without loss of generality we assume that the cyclic group G underlying \mathcal{A} and \mathcal{A}' is minimal possible. Then $G \neq 1$. Moreover,

$$(21) \quad U \in \mathcal{G}(\mathcal{A}) \ \& \ U \neq G \implies U \cap H = 1$$

and

$$(22) \quad L \in \mathcal{G}(\mathcal{A}) \ \& \ L \neq 1 \implies LH = G.$$

Indeed, relation (21) follows from relation (22) by duality. To prove (22) suppose on the contrary that $LH \neq G$. Since L and H are \mathcal{A}' -groups, so is the group LH . By the minimality of G we have $\mathcal{G}(\mathcal{A}_{G/L}) = \mathcal{G}(\mathcal{A}'_{G/L})$, and hence LH is an \mathcal{A} -group. Again by the minimality this implies that $\mathcal{G}(\mathcal{A}'_{LH}) = \mathcal{G}(\mathcal{A}_{LH})$. It follows that H is an \mathcal{A} -group. Contradiction.

By the quasidensity of \mathcal{A} there exist \mathcal{A} -groups U and L such that the numbers $|G/U|$ and $|L|$ are prime. Since $1 < H < G$, from (21) and (22) it follows that $G = U \times H = L \times H$. Thus the numbers $|H|$ and $|G/H|$ are prime. Therefore the S-ring \mathcal{A} is elementary nondense. By the lemma hypothesis this implies that so is \mathcal{A}' . This implies that $H \notin \mathcal{G}(\mathcal{A}')$ in contrast to the choice of H . ■

7. LIFTING: DENSE CASE

In this section we are to get an analog of the theory developed in Section 6 but this time for dense S-rings. In what follows under a p -section we mean a section which is a p -group.

Theorem 7.1. *Let \mathcal{A} be a circulant dense S-ring. Then $\text{rad}(\mathcal{A}) \neq 1$ if and only if \mathcal{A} contains a non-quasisubprinciple p -section.*

Proof. The “if” part is obvious. To prove the “only if” part suppose that $\text{rad}(\mathcal{A}) \neq 1$. Then by [4, Theorem 5.4] the S-ring \mathcal{A} is a U/L -wreath product such that

$$(23) \quad |G/U| = |L| = p$$

where p is a prime and G is the underlying group of \mathcal{A} . To complete the proof it suffices to verify that the \mathcal{A} -section G_p is non-quasisubprinciple. Suppose on the

contrary that G_p is a quasisubsection of a principal \mathcal{A} -section $T = H/K$. Then $|T_p| = |G_p|$ and hence

$$(24) \quad K_p = (G/H)_p = 1.$$

On the other hand, since \mathcal{A} is the U/L -wreath product, Theorem 4.3 implies that the S-ring \mathcal{A}_T is a U_T/L_T -wreath product where

$$U_T/L_T = \rho_{G,T}(U/L) = (U \cap H)K/(L \cap H)K.$$

Using (23) and (24) we obtain by comparing p -parts that $(U \cap H)K < H$ and $(L \cap H)K > K$. It follows that $U_T \neq T$ and $L_T \neq 1$. Therefore \mathcal{A}_T is a proper generalized wreath product. However, this is impossible because the section T is principle and hence $\text{rad}(\mathcal{A}_T) = 1$. ■

As the following example shows, not every minimal non-quasisubprinciple section is a p -section.

Example. Let p, q, r be distinct primes such that $r - 1$ is divided by both p and q . Let us define S-rings \mathcal{A}_1 and \mathcal{A}_2 over the group $\mathbb{Z}_{rp^2q^2}$ as follows

$$\mathcal{A}_1 = \text{Cyc}(K_{1,1}, \mathbb{Z}_{rp^2}) \otimes \text{Cyc}(K_{1,2}, \mathbb{Z}_{q^2}),$$

$$\mathcal{A}_2 = \text{Cyc}(K_{2,1}, \mathbb{Z}_{p^2}) \otimes \text{Cyc}(K_{2,2}, \mathbb{Z}_{rq^2})$$

where $K_{1,1}$, $K_{1,2}$, $K_{2,1}$ and $K_{2,2}$ are groups of order p , q , p and q respectively. The groups $K_{1,2}$ and $K_{2,1}$ are uniquely determined; choose the groups $K_{1,1}$ and $K_{2,2}$ so that none of the coordinate projections is trivial. Then $\text{Cyc}(K_{1,1}, \mathbb{Z}_{rp^2})$ and $\text{Cyc}(K_{2,1}, \mathbb{Z}_{rq^2})$ are S-rings with trivial radicals. Moreover, using natural identifications we have

$$(\mathcal{A}_1)^{p^2q^2} = \text{Cyc}(K_{2,1}, \mathbb{Z}_{p^2}) \otimes \text{Cyc}(K_{1,2}, \mathbb{Z}_{q^2}) = (\mathcal{A}_2)_{p^2q^2}$$

where $(\mathcal{A}_1)^{p^2q^2}$ is the restriction of \mathcal{A}_1 to the factorgroup of order p^2q^2 whereas $(\mathcal{A}_2)_{p^2q^2}$ is the restriction of \mathcal{A}_2 to the subgroup of order p^2q^2 . Then one can form the generalized wreath product

$$\mathcal{A} = \mathcal{A}_1 \wr_{U/L} \mathcal{A}_2$$

over the group $\mathbb{Z}_{p^2q^2r^2}$ where U and L are the subgroups of this group of index and order r respectively. It is not difficult to verify that U/L is a minimal non-quasisubprinciple \mathcal{A} -section.

Now let \mathcal{A} be a quasidense S-ring over a cyclic group G and S_0 an \mathcal{A} -section. Suppose that $\text{rad}(\mathcal{A}_{S_0}) > 1$ and S_0 is a p -section where p is a prime divisor of $|G|$. Then the S-ring \mathcal{A}_{S_0} is the U_0/L_0 -wreath product where U_0 and L_0 are subgroups of S_0 of index and order p respectively. Next, the sets

$$\{U \in \mathcal{G}(\mathcal{A}) : \rho_{G,S_0}(U) = U_0\} \quad \text{and} \quad \{L \in \mathcal{G}(\mathcal{A}) : \rho_{G,S_0}(L) = L_0\}$$

are nonempty, because they contain the groups $\pi^{-1}(U_0)$ and $\pi^{-1}(L_0)$ respectively, where $\pi = \pi_{S_0}$. Moreover, since the lattice $\mathcal{G}(\mathcal{A})$ is distributive [16, p.11], these sets have the greatest and least elements. Denote them by $U = U(S_0)$ and $L = L(S_0)$, respectively. Clearly, $U \geq L$.

The definition of the section U/L associated with S_0 is uniform. Namely, let S be an \mathcal{A} -section that contains S_0 as a subsection. Since the mapping $\rho_{G,S}$ defined in Subsection 4.1 induces a lattice epimorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{A}_S)$, we have

$$(25) \quad \rho_{G,S}(U/L) = U_S/L_S$$

where U_S/L_S is the section of the S-ring \mathcal{A}_S defined by S_0 in the same way as the section U/L in the S-ring \mathcal{A} . In particular, $U_{S_0}/L_{S_0} = U_0/L_0$.

Lemma 7.2. *In the above notation let $M/N \in \mathfrak{S}(\mathcal{A})$ be such that $M \not\leq U$ and $N \not\leq L$. Then S_0 is an \mathcal{A} -quasisubsection of M/N .*

Proof. Let $S_0 = V/K$. Then by the definition of U and L we have

$$U_0 = (U \cap V)K/K \quad \text{and} \quad L_0 = (L \cap V)K/K.$$

Since S_0 is a cyclic p -group, this implies that $|V_p : U_p| = p = |L_p : K_p|$. On the other hand, from the hypothesis of the lemma it follows by the maximality of U that $M_p > U_p$, and by the minimality of L that $N_p < L_p$. Thus $V_p \leq M_p$ and $N_p \leq K_p$. Therefore

$$V_p N_{p'} \leq M_p M_{p'} \leq M \quad \text{and} \quad K_p N_{p'} \geq N_p N_{p'} \geq N.$$

Thus $T = V_p N_{p'} / K_p N_{p'}$ is an $\mathcal{A}_{M/N}$ -section. Obviously, T is a multiple of V_p / K_p . Since the latter section is projectively equivalent to S_0 , we are done. ■

From Theorem 5.2 it follows that any circulant S-ring with nontrivial radical, is a proper generalized wreath product. Statement (1) of the following theorem gives an explicit form of such a product in the quasidense case (cf. Theorem 7.1).

Theorem 7.3. *Let \mathcal{A} be a quasidense S-ring over a cyclic group G . Suppose that $S_0 \in \mathfrak{S}(\mathcal{A})$ is a non-quasiprincipal p -section. Then*

- (1) \mathcal{A} is a proper U/L -wreath product where $U = U(S_0)$ and $L = L(S_0)$,
- (2) if $T \in \mathfrak{S}(\mathcal{A})$ is a subsection of neither U nor G/L , then S_0 is a quasisubsection of T .

Proof. To prove the first statement it suffices to verify that $\text{rad}(X) \geq L$ for all $X \in \mathcal{S}(\mathcal{A})_{G \setminus U}$. Suppose on the contrary that this is not true for some X . Then S_0 is a quasisubsection of the section M/N where $M = \langle X \rangle$ and $N = \text{rad}(X)$ (Lemma 7.2). However, this contradicts the theorem hypothesis because the section M/N is principal. To prove the second statement suppose that an \mathcal{A} -section $T = M/N$ is a subsection of neither U nor G/L . Then $M \not\leq U$ and $N \not\leq L$. Thus the required statement immediately follows from Lemma 7.2. ■

8. COSET S-RINGS

8.1. Definition and basic properties. In this section we introduce and study circulant coset S-rings. In a sense these rings are antipodes of rational circulant S-rings. Indeed, as we will see below (Theorems 8.3 and 8.4) an S-ring is a coset one if and only if it can be constructed from group rings by tensor and generalized wreath products,⁶ whereas an S-ring is rational if and only if it can be constructed from S-rings of rank 2 in the same way (the latter follows from [11, Theorem 1.2]).

Definition 8.1. *An S-ring over an abelian group G is a coset one, if any of its basic sets is a coset of a subgroup in G .*

By definition any basic set X of a coset S-ring \mathcal{A} is of the form $X = xH$ for some group $H \leq G$ and $x \in X$. It is easily seen that $H = \text{rad}(X)$, and hence

$$(26) \quad X = x \text{rad}(X)$$

⁶In fact, the tensor product here is needless.

for any $x \in X$. However, $\text{rad}(X)$ is an \mathcal{A} -group. Thus, any basic set of \mathcal{A} is a coset of a uniquely determined \mathcal{A} -group.

The following statement expressing the “radical monotony property” of a coset circulant S-ring, will be used below.

Lemma 8.2. *Let \mathcal{A} be a circulant coset S-ring. Then given $X, Y \in \mathcal{S}(\mathcal{A})$, the inclusion $Y \subset \langle X \rangle$ implies $\text{rad}(Y) \leq \text{rad}(X)$.*

Proof. Let $X, Y \in \mathcal{S}(\mathcal{A})$. Then from (26) it follows that X and Y are cosets of the groups $\text{rad}(X)$ and $\text{rad}(Y)$ respectively. Therefore the set X^π where $\pi = \pi_{G/\text{rad}(X)}$, is a singleton consisting of a generator of the group $S = \langle X \rangle / \text{rad}(X)$. This implies that $\mathcal{A}_S = \mathbb{Z}S$. If $Y \leq \langle X \rangle$, then Y^π is a basic set of \mathcal{A}_S , and hence $Y^\pi \subset S$ is also a singleton. It follows that $\text{rad}(Y)^\pi \leq \text{rad}(Y)^\pi = 1$. Thus $\text{rad}(Y) \leq \text{rad}(X)$ as required. ■

The circulant coset S-rings can be characterized in terms of their sections as follows.

Theorem 8.3. *For a circulant S-ring \mathcal{A} the following statements are equivalent:*

- (1) \mathcal{A} is a coset S-ring,
- (2) $\mathcal{A}_S = \mathbb{Z}S$ for any principal \mathcal{A} -section S ,
- (3) $\mathcal{A}_S = \mathbb{Z}S$ for any \mathcal{A} -section S with trivial radical,
- (4) $\mathcal{A}_S = \mathbb{Z}S$ for any quasisubprincipal \mathcal{A} -section S .

Proof. Statements (1) and (2) are equivalent: implication (1) \Rightarrow (2) follows from (26) whereas implication (2) \Rightarrow (1) follows from the definition of principal section. Next, implication (4) \Rightarrow (2) is obvious and implication (2) \Rightarrow (4) is true because the equality in statement (4) is preserved under projective equivalence and taking subsections. Finally, any principal \mathcal{A} -section obviously has trivial radical, and any \mathcal{A} -section with trivial radical is subprincipal (Lemma 5.3). Thus implications (3) \Rightarrow (2) and (4) \Rightarrow (3) hold. ■

Any primitive section S of a dense circulant S-ring \mathcal{A} has prime order. Therefore $\text{rad}(\mathcal{A}_S) = 1$. Thus if \mathcal{A} is a coset S-ring, then $\mathcal{A}_S = \mathbb{Z}S$ for any primitive \mathcal{A} -section S (Theorem 8.3). The converse statement is not true, a counterexample is given by $\mathcal{A} = \text{Cyc}(\{\pm 1\}, \mathbb{Z}_8)$.

Theorem 8.4. *The class of circulant coset S-rings is closed under restriction to a section and under tensor and generalized wreath products, and consists of quasidense S-rings.*

Proof. Since any quotient epimorphism takes a coset to a coset, and the product of cosets is a coset, the closedness statement follows from the definitions of tensor and generalized wreath products. The quasidensity statement is true because any non-quasidense S-ring has a rank 2 section of composite order (Theorem 5.5) whereas by above no coset S-ring can have such a section. ■

The intersection of circulant coset S-rings is not necessarily a coset one: a counterexample is given by the S-ring $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$ over the group \mathbb{Z}_{pq} where

$$\mathcal{A}_1 = \mathbb{Z}\mathbb{Z}_p \wr \mathbb{Z}\mathbb{Z}_q \quad \text{and} \quad \mathcal{A}_2 = \mathbb{Z}\mathbb{Z}_q \wr \mathbb{Z}\mathbb{Z}_p$$

with p and q distinct primes. One can see that its rank equals 2, and hence it is not coset. Moreover, \mathcal{A} is even not quasidense (Theorem 5.5). The following statement shows that this is a unique obstacle.

Theorem 8.5. *The intersection of circulant coset S-rings is a coset one whenever it contains a quasidense S-ring.*

Proof. Let $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$ where \mathcal{A}_1 and \mathcal{A}_2 are circulant coset S-rings. Suppose that \mathcal{A} contains a quasidense S-ring. Then by Theorem 5.6 we can assume that \mathcal{A} is quasidense. By implication (3) \Rightarrow (1) of Theorem 8.3 it suffices to verify that $\mathcal{A}_S = \mathbb{Z}S$ for an \mathcal{A} -section S with trivial radical. However, since $\mathcal{A}_i \geq \mathcal{A}$ for $i = 1, 2$, any such section is an \mathcal{A}_i -section and $(\mathcal{A}_i)_S \geq \mathcal{A}_S$. The quasidensity of the S-ring \mathcal{A}_S implies by Theorem 5.7 that $\text{rad}((\mathcal{A}_i)_S) = 1$. Taking into account that \mathcal{A}_i is a coset S-ring, we conclude that $(\mathcal{A}_i)_S = \mathbb{Z}S$ (implication (1) \Rightarrow (3) of Theorem 8.3). Thus by (6) we have

$$\mathcal{A}_S = (\mathcal{A}_1)_S \cap (\mathcal{A}_2)_S = \mathbb{Z}S$$

as required. ■

In what follows any representation of an S-ring \mathcal{A} as a proper generalized wreath product, will be called a *gwr-decomposition* of \mathcal{A} . Given an \mathcal{A} -section S we say that the T' -decomposition of \mathcal{A}_S is *lifted* to a T -decomposition of \mathcal{A} if \mathcal{A} is the T -wreath product and $T_S = T'$ (cf. Theorem 4.3).

Theorem 8.6. *Let S be a section of a circulant coset S-ring \mathcal{A} . Then any gwr-decomposition of the S-ring \mathcal{A}_S can be lifted to a gwr-decomposition of \mathcal{A} .*

Proof. Let \mathcal{A}_S be a proper V/K -wreath product. To lift it to a gwr-decomposition of \mathcal{A} it suffices to consider two cases depending on whether S is a subgroup or quotient of G . By duality the latter case follows from the former one by statement (2) of Lemma 4.4 and statement (3) of Theorem 2.3. In the former case V and K are also subgroups of G . Set $L = K$ and

$$U = \langle \{X \in \mathcal{S}(\mathcal{A}) : \langle X \rangle \cap S \leq V\} \rangle.$$

Clearly, $U \in \mathcal{G}(\mathcal{A})$ and $U \geq V \geq L$. On the other hand, if H_1 and H_2 are subgroups of G such that $H_1 \cap S \leq V$ and $H_2 \cap S \leq V$, then by the distributivity of the lattice $\mathcal{G}(\mathcal{A})$ we have

$$H_1 H_2 \cap S = (H_1 \cap S)(H_2 \cap S) \leq V.$$

This shows that $U \cap S \leq V$, and hence $U \cap S = V$. In particular, $U \neq G$. To complete the proof let X be a basic set of \mathcal{A} outside U . Then by the definition of U the group $H = \langle X \rangle \cap S$ is not a subgroup of V . Since \mathcal{A}_S is the V/K -wreath product, this implies that $K \leq \text{rad}(Y)$ where Y is a highest basic set of \mathcal{A}_H . However, then by Lemma 8.2 we have

$$L = K \leq \text{rad}(Y) \leq \text{rad}(X).$$

Thus \mathcal{A} is a proper U/L -wreath product. Since also $\rho_{G,S}(U/L) = V/K$, we are done. ■

8.2. Elementary coset S-rings. From Theorems 8.3 and 8.4 it follows that any circulant coset S-ring can be constructed from group rings by generalized wreath products. In the rest of this section we are interested in the coset S-rings that are obtained in one iteration of the above process. More precisely, by Theorem 2.1 given a section $T = U/L$ of a cyclic group G one can form the S-ring

$$(27) \quad \mathbb{Z}(G, T) = \mathbb{Z}U \wr_T \mathbb{Z}(G/L)$$

because any group ring is dense and the restrictions of both $\mathbb{Z}U$ and $\mathbb{Z}(G/L)$ to T equal $\mathbb{Z}T$. It is easily seen that $\mathbb{Z}(G, T)$ is a coset S-ring over G .

Definition 8.7. Any S-ring of the form (27) is called *elementary coset*.

Clearly, the group ring $\mathbb{Z}G$ is elementary coset (in this case generalized wreath product (27) is not proper). It is also easily seen that every basic set of elementary coset S-ring (27) inside U is a singleton whereas the basic sets outside U are L -cosets. Any elementary coset S-ring is schurian (Theorem 3.6), and the automorphism group of it is the canonical generalized wreath product of U_{right} by $(G/L)_{right}$ in the sense of [9]. For association schemes a similar situation was studied in [15].

Let \mathcal{A} be elementary coset S-ring (27). Given a function $t \in L^{G/U}$ and an element $g \in G$ we define a permutation $\sigma_{t,g} \in \text{Sym}(G)$ by

$$(28) \quad \sigma_{t,g} : x \mapsto xt(X)g, \quad x \in G,$$

where X is the U -coset containing x . The set of all these permutations forms a group with identity $\sigma_{1,1}$ where the first 1 in subscript denotes the function taking every x to 1, and the multiplication satisfying

$$(29) \quad \sigma_{t_1,g_1}\sigma_{t_2,g_2} = \sigma_{t_1t_2,g_1g_2}$$

for all $t_1, t_2 \in L^{G/U}$ and $g_1, g_2 \in G$. Clearly, $\{\sigma_{1,g} : g \in G\} = G_{right}$ and $\sigma_{1,g} = \sigma_{t,1}$ for all $g \in L$ where t is the function taking every x to g .

It follows from [8, Theorem 7.2] that $\sigma_{t,g} \in \text{Aut}(\mathcal{A})$ whenever $t(U) = g = 1$. Since $G_{right} \leq \text{Aut}(\mathcal{A})$, the permutation $\sigma_{t,g}$ is an automorphism of \mathcal{A} for all t and g . In fact, statement (2) of the theorem below shows that \mathcal{A} has no other automorphisms.

Theorem 8.8. Let $\mathcal{A} = \mathbb{Z}(G, T)$ be an elementary coset S-ring (27) and S an \mathcal{A} -section. Then

- (1) $\mathcal{A}_S = \mathbb{Z}(S, T_S)$ where T_S is the section defined in (17),
- (2) $\text{Aut}(\mathcal{A}) = \{\sigma_{t,g} : t \in L^{G/U}, g \in G\}$,
- (3) $\text{Aut}(\mathcal{A})^S = \text{Aut}(\mathcal{A}_S)$.

Proof. By Theorem 4.3 the S-ring \mathcal{A}_S is the T_S -wreath product of the S-rings \mathcal{A}_{U_S} and $\mathcal{A}_{S/L_S} = \mathcal{A}_{(G/L)_S}$ which are Cayley isomorphic by Lemma 4.1 and Theorem 4.2 to the S-rings \mathcal{A}_{S_U} and $\mathcal{A}_{S_{G/L}}$ respectively. Since $\mathcal{A}_U = \mathbb{Z}U$ and S_U is an \mathcal{A}_U -section, as well as $\mathcal{A}_{G/L} = \mathbb{Z}(G/L)$ and $S_{G/L}$ is an $\mathcal{A}_{G/L}$ -section, we have $\mathcal{A}_{S_U} = \mathbb{Z}S_U$ and $\mathcal{A}_{S_{G/L}} = \mathbb{Z}(S_{G/L})$ which proves statement (1).

To prove statement (2) denote by Γ the group in the right-hand side of the equality, Then from the discussion before the theorem it follows that $\Gamma \leq \text{Aut}(\mathcal{A})$. Therefore to check the reverse inclusion it suffices to prove that $|\text{Aut}(\mathcal{A})| \leq |\Gamma|$. However, since $\mathcal{A}_{G/L} = \mathbb{Z}(G/L)$, we have

$$\text{Aut}(\mathcal{A})^{G/L} = (G/L)_{right} = \Gamma^{G/L}.$$

To complete the proof we show that the kernel of the epimorphism

$$\pi : \text{Aut}(\mathcal{A}) \rightarrow \text{Aut}(\mathcal{A})^{G/L}$$

is contained in Γ , more precisely that any $\sigma \in \ker(\pi)$ is of the form $\sigma_{t,1}$ for some $t \in L^{G/U}$. Note that such a permutation σ leaves each $X \in G/U$ fixed as a set. Therefore all we need to prove is that σ^X acts on X by multiplying by an element

$l = l(X)$ belonging to L . Let $g_X \in G_{U \rightarrow X}$. Then obviously the permutation $\sigma' = g_X \sigma g_X^{-1}$ belongs to $\ker(\pi)$ and leaves the set U fixed. Taking into account that $\mathcal{A}_U = \mathbb{Z}U$, we see that $(\sigma')^U \in \text{Aut}(\mathcal{A})^U = U_{\text{right}}$. Therefore since σ' leaves also any L -coset fixed, there exists $l \in L$ for which

$$u^{g_X \sigma g_X^{-1}} = u^{\sigma'} = lu, \quad u \in U.$$

When u runs over U , the element $x = u^{g_X}$ runs over X . Therefore the above equality implies that $x^\sigma = lx$ for all $x \in X$, as required.

To prove statement (3) we observe that obviously $\text{Aut}(\mathcal{A})^S \leq \text{Aut}(\mathcal{A}_S)$. To verify the converse inclusion, let $f' \in \text{Aut}(\mathcal{A}_S)$ and $S = H/K$. Then by statements (1) and (2) we have $f' = \sigma_{t',g'}$ where $t' \in L_S^{S/U_S}$ and $g' \in S$. However, from (29) it follows that $\sigma_{t',g'} = \sigma_{t',1} \sigma_{1,g'}$. Moreover, the permutation $\sigma_{1,g'} \in S_{\text{right}}$ can be lifted to a permutation in G_{right} , because obviously $\Gamma^S = S_{\text{right}}$ where Γ is the setwise stabilizer of H in G_{right} . Thus we can assume that $f' = \sigma_{t',1}$.

To verify that f' can be lifted to a permutation in $\text{Aut}(\mathcal{A})$ it suffices to check that there exists $t \in L^{G/U}$ such that

$$(30) \quad (\sigma_{t,1})^S = \sigma_{t',1}.$$

To define the function t given $X \in G/U$ set $t(X) = 1$ if the set $X \cap H$ is empty. Suppose that it is not empty. Then $X \cap H$ is a coset of $H \cap U$. Recall that

$$U_S = (U \cap H)K/K \quad \text{and} \quad L_S = (L \cap H)K/K.$$

Therefore, the first equality implies that $X' := \pi(X \cap H)$ belongs to S/U_S where $\pi = \pi_S$. Moreover, the set $L \cap \pi^{-1}(t'(X'))$ is not empty by the second equality. Now set $t(X)$ to be any element of that set. It follows that in any case $t \in L^{G/U}$, and the permutation $\sigma_{t,1}$ leaves H fixed. Thus, given $x' \in S$ we have

$$(x')^{\sigma_{t,1}^S} = x' \pi(t(X)) = x' t'(X') = (x')^{\sigma_{t',1}}$$

where X' is the U_S -coset containing x' , which proves (30). ■

In general, an extension of a coset S-ring is not necessary coset: a non-coset S-ring $\text{Cyc}(\{\pm 1\}, \mathbb{Z}_8)$ contains a coset S-ring $\text{Cyc}(K, \mathbb{Z}_8)$ where $K = \text{Aut}(\mathbb{Z}_8)$. However, we can prove the following auxiliary statement to be used in Section 9.

Lemma 8.9. *Let \mathcal{A} be an elementary coset S-ring (27). Suppose that either $|L|$ or $|G/U|$ is prime. Then any extension of \mathcal{A} is an elementary coset S-ring.*

Proof. By duality (Theorem 2.3) without loss of generality we can assume that $|L|$ is prime. Let $\mathcal{A}' \geq \mathcal{A}$. Denote by U' the group consisting of all $x \in G$ for which $\{x\}$ is a basic set of \mathcal{A}' . Clearly, U' is an \mathcal{A}' -group containing U . We claim that any basic set not in U' , is a basic set of \mathcal{A} . Then since $\mathcal{A}' \geq \mathcal{A}$, we have $\mathcal{A}' = \mathbb{Z}(G, T')$ where $T' = U'/L$, as required.

Suppose that the claim is not true. Then there exists a basic set $X \not\subset U'$ of \mathcal{A} , that is not a basic set of \mathcal{A}' . It follows that $X \subset G \setminus U$. Since $\mathcal{A} = \mathbb{Z}(T, G)$, we have

$$(31) \quad X = xL$$

for all $x \in X$. Moreover, X is the union of at least two basic sets of \mathcal{A}' . If one of them, say X' , has nontrivial radical, then the latter coincides with L , because $\mathcal{A}'_{G/L} \geq \mathcal{A}_{G/L} = \mathbb{Z}(G/L)$. But then from (31) it follows that

$$|L| = |X| \geq |X'| \geq |L|.$$

Therefore $X = X'$. Contradiction.

Now we have $\text{rad}(X') = 1$ for all $X' \in \mathcal{S}(\mathcal{A}')_X$. Since the group $H := \langle X \rangle$ is cyclic, there exists such an X' such that $\langle X' \rangle = H$. Therefore $\text{rad}(\mathcal{A}'_H) = 1$. On the other hand, since $\mathcal{A}_H \leq \mathcal{A}'_H$ and \mathcal{A} is quasidense, we conclude by Theorem 5.6, then \mathcal{A}'_H is also quasidense. Thus by Theorem 5.7 we have

$$(32) \quad \mathcal{A}'_H = \text{Cyc}(K, H)$$

for some group $K \leq \text{Aut}(H)$. Clearly, K is a subgroup of the stabilizer of 1 in $\text{Aut}(\mathcal{A}_H)$. Since this stabilizer is a p -group (statements (1) and (2) of Theorem 8.8) where $p = |L|$, it follows that K is also a p -group. This implies that the cardinality of every K -orbit is a p th power. Since X is a K -invariant set of cardinality p , equality (32) implies that either every element in $\mathcal{S}(\mathcal{A}')_X$ is singleton, or $\mathcal{S}(\mathcal{A}')_X = \{X\}$. In both cases we come to a contradiction with the choice of X . ■

9. SCHURITY AND SEPARABILITY OF COSET S-RINGS

The theory of coset circulant S-rings developed in Section 8 enables us to prove the following theorem which is the main result of the section.

Theorem 9.1. *Any circulant coset S-ring is schurian and separable.*

We will deduce Theorem 9.1 in the end of the section from two following auxiliary statements on the automorphism group of a coset S-ring. In what follows, if an S-ring \mathcal{A} is the S -wreath product where $S \in \mathfrak{S}(\mathcal{A})$, then we say that S is a *gwr-section* of \mathcal{A} .

Theorem 9.2. *Let \mathcal{A} be a coset S-ring over a cyclic group G . Then the group $\text{Aut}(\mathcal{A})$ is generated by the automorphism groups of elementary coset S-rings $\mathbb{Z}(G, S)$ where S runs over all gwr-sections of \mathcal{A} .*

Proof. Induction on the order of G . The base case of the induction is obvious. Let $G \neq 1$ and $f \in \text{Aut}(\mathcal{A})$. By Theorem 8.4 the S-ring \mathcal{A} is quasidense. So there exists an \mathcal{A} -group L of prime order. Set $S = G/L$. Since $f^S \in \text{Aut}(\mathcal{A}_S)$, by the inductive hypothesis f^S is the product of automorphisms of elementary coset S-rings $\mathbb{Z}(S, T')$ where T' runs over all gwr-sections of \mathcal{A}_S . On the other hand, by Theorem 8.6 any T' -decomposition of \mathcal{A}_S can be lifted to a T -decomposition of \mathcal{A} . Set $\mathcal{B} = \mathbb{Z}(G, T)$. Then $\mathcal{B}_S = \mathbb{Z}(S, T')$ by statement (1) of Theorem 8.8. Moreover, by statement (3) of that theorem we also have $\text{Aut}(\mathcal{B})^S = \text{Aut}(\mathcal{B}_S)$. Thus to write f as the product of automorphisms of elementary coset S-rings, without loss of generality we can assume that

$$(33) \quad f^S = \text{id}_S.$$

Denote by \mathcal{A}' the S-ring over G associated with the group $\Gamma = \langle G_{\text{right}}, f \rangle$. In particular, $f \in \text{Aut}(\mathcal{A}')$. Due to (33) we have $\Gamma^S = S_{\text{right}}$. Therefore

$$(34) \quad \mathcal{A}'_{G/L} = \mathbb{Z}(G/L).$$

Moreover, $\mathcal{A}' \geq \mathcal{A}$ because $f \in \text{Aut}(\mathcal{A})$. It follows that $\mathcal{A}'_L \geq \mathcal{A}_L$. On the other hand, \mathcal{A}_L is a coset S-ring (Theorem 8.4) with trivial radical because $|L|$ is prime. Thus $\mathcal{A}_L = \mathbb{Z}L$, and hence $\mathcal{A}'_L = \mathbb{Z}L$. Together with (34) this implies that \mathcal{A}' is an extension of elementary coset S-ring $\mathbb{Z}(G, L/L)$. Therefore by Lemma 8.9 we conclude that

$$(35) \quad \mathcal{A}' = \mathbb{Z}(G, T')$$

where $T' = U'/L$ with U' being the largest \mathcal{A}' -group for which $\mathcal{A}'_{U'} = \mathbb{Z}U'$.

Denote by U the largest \mathcal{A} -subgroup of G inside U' . Clearly, $L \leq U$. Take a set $X \in \mathcal{S}(\mathcal{A})$ outside U . Then by the definition of U there exists a basic \mathcal{A}' -set $X' \subset X$ that is outside U' . By (35) this implies that X' is an L -coset. However, X is a coset of $\text{rad}(X)$ because \mathcal{A} is a coset S-ring. Thus $L \leq \text{rad}(X)$. This proves that $T := U/L$ is a gwr-section of \mathcal{A} . Besides, since $U \leq U'$ we have

$$\mathbb{Z}(G, T) \leq \mathbb{Z}(G, T').$$

Since $f \in \text{Aut}(\mathcal{A}')$, this inclusion together with (35) implies that f is an automorphism of the elementary coset S-ring $\mathbb{Z}(G, T)$, as required. ■

Theorem 9.3. *Let \mathcal{A} be a circulant coset S-ring. Then $\text{Aut}(\mathcal{A})^S = \text{Aut}(\mathcal{A}_S)$ for any \mathcal{A} -section S .*

Proof. Obviously, $\text{Aut}(\mathcal{A})^S \leq \text{Aut}(\mathcal{A}_S)$. To prove the reverse inclusion we observe that by Theorem 9.2 each automorphism of \mathcal{A}_S can be written as the product of automorphisms of elementary coset S-rings $\mathbb{Z}(S, T')$ where T' runs over all gwr-sections of \mathcal{A}_S . On the other hand, by Theorem 8.6 any T' -decomposition of \mathcal{A}_S can be lifted to a T -decomposition of \mathcal{A} . Thus the required statement follows from statement (3) of Theorem 8.8. ■

Proof of Theorem 9.1. Let \mathcal{A} be a coset S-ring over a cyclic group G . We will prove both statements by induction on the order of this group. Let $|G| > 1$. If $\text{rad}(\mathcal{A}) = 1$, then $\mathcal{A} = \mathbb{Z}G$ by implication (1) \Rightarrow (3) of Theorem 8.3 and the statements are obvious. Let now $\text{rad}(\mathcal{A}) > 1$. Then by statement (1) of Theorem 5.2 there is a gwr-decomposition

$$(36) \quad \mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L}.$$

By Theorem 8.4 the S-rings \mathcal{A}_U and $\mathcal{A}_{G/L}$ are coset ones. Thus they are schurian and separable by the inductive hypothesis.

To prove that \mathcal{A} is schurian set $\Delta_1 = \text{Aut}(\mathcal{A}_U)$ and $\Delta_0 = \text{Aut}(\mathcal{A}_{G/L})$. Then by Theorem 3.6 it suffices to verify that $(\Delta_1)^{U/L} = (\Delta_0)^{U/L}$. However, from Theorem 9.3 it follows that

$$(\Delta_1)^{U/L} = \text{Aut}(\mathcal{A}_U)^{U/L} = \text{Aut}(\mathcal{A}_{U/L}) = \text{Aut}(\mathcal{A}_{G/L})^{U/L} = (\Delta_0)^{U/L}$$

as required.

To prove that \mathcal{A} is separable let $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$ be a similarity. Then by statement (1) of Theorem 3.3 the S-ring \mathcal{A}' is the U'/L' -wreath product where $U' = U^\varphi$ and $L' = L^\varphi$. By the inductive hypothesis the similarities

$$\varphi_U : \mathcal{A}_U \rightarrow \mathcal{A}'_{U'} \quad \text{and} \quad \varphi_{G/L} : \mathcal{A}_{G/L} \rightarrow \mathcal{A}'_{G'/L'}$$

are induced by some bijections, say f_1 and f_0 . Given $X \in G/U$ let us fix two bijections $g \in G_{U \rightarrow X}$ and $g' \in G'_{X' \rightarrow U'}$. By Lemma 3.4 the bijection $g^{U/L}(f_0)^{X/L}(g')^{X'/L'}$ induces the similarity $\varphi^{U/L}$. It follows that

$$(37) \quad g^{U/L}(f_0)^{X/L}(g')^{X'/L'}((f_1)^{U/L})^{-1} \in \text{Aut}(\mathcal{A}_{U/L}).$$

By Theorem 9.3 there exists an automorphism $h_X \in \text{Aut}(\mathcal{A}_U)$ such that its restriction to U/L coincides with the automorphism in the left-hand side of (37). Now, the family of bijections

$$f_X := g^{-1}h_X f_1(g')^{-1}, \quad X \in G/U,$$

defines a bijection $f : G \rightarrow G'$ such that $(G/U)^f = G'/U'$ and $(G/L)^f = G'/L'$. Then obviously

$$(38) \quad gf^X g' = g f_X g' = h_X f_1.$$

By (12) this implies that the bijection $gf^X g'$ induces the similarity φ_U for all X . This proves the second part of condition (13). Let us check the first part of that condition. For any $X \in G/U$ from the definition of h_X and (38) it follows that

$$\begin{aligned} f^{X/L} &= (g^{-1}h_X f_1(g')^{-1})^{X/L} = \\ &= (g^{U/L})^{-1} g^{U/L} (f_0)^{X/L} (g')^{X'/L'} ((f_1)^{U/L})^{-1} (f_1)^{U/L} ((g')^{X'/L'})^{-1} = (f_0)^{X/L}. \end{aligned}$$

Therefore $f^{G/L} = f_0$, and hence the bijection $f^{G/L}$ induces the similarity $\varphi_{G/L}$. Thus by Theorem 3.5 the bijection f induces the similarity φ as required. ■

The following statement can in a sense be regarded as a combinatorial analog of Theorem 9.1.

Corollary 9.4. *Any circulant coset S-ring is an intersection of elementary coset S-rings over the same group.*

Proof. Let \mathcal{A} be a circulant coset S-ring. Denote by \mathcal{A}' the intersection of all elementary coset S-rings $\mathbb{Z}(G, S)$ where S runs over all gwr-sections of \mathcal{A} . Since \mathcal{A} is quasidense (Theorem 8.4), the S-ring \mathcal{A}' is coset (Theorem 8.5). So from Theorem 9.1 it follows that it is schurian. Therefore \mathcal{A}' equals the S-ring associated with the group Γ generated by the automorphism groups of the above S-rings $\mathbb{Z}(G, S)$. On the other hand, by Theorem 9.1 the S-ring \mathcal{A} is also schurian. By Theorem 9.2 this implies that \mathcal{A} equals the S-ring associated with the group Γ . Thus $\mathcal{A} = \mathcal{A}'$. ■

10. COSET CLOSURE

10.1. Relative coset closure. We start with developing a technique to find the schurian closure of a quasidense circulant S-ring \mathcal{A} (Theorem 1.2). The key point of this technique is its coset closure \mathcal{A}_0 defined in Introduction (Definition 1.1) as the intersection of all coset S-rings containing \mathcal{A} . By Theorem 8.5 the S-ring \mathcal{A}_0 is a coset one. Moreover,

$$(39) \quad (\mathcal{A}_0)_S = \mathbb{Z}S, \quad S \in \mathfrak{S}_0(\mathcal{A})$$

where $\mathfrak{S}_0(\mathcal{A})$ is the class of all quasisubprincipal \mathcal{A} -sections. Indeed, by Theorem 4.2 this is reduced to the case of a principal S , in which the required statement follows from Theorem 5.7 and implication (1) \Rightarrow (3) of Theorem 8.3. Thus the S-ring \mathcal{A}_0 equals the intersection of all coset S-rings \mathcal{A}' such that

$$(40) \quad \mathcal{A}' \geq \mathcal{A} \quad \text{and} \quad (\mathcal{A}')_S = \mathbb{Z}S$$

for all $S \in \mathfrak{S}_0(\mathcal{A})$. For induction reasons it is convenient to generalize the coset closure concept by permitting the section S to run over a larger class \mathfrak{S} .

Definition 10.1. *A class $\mathfrak{S} \subset \mathfrak{S}(\mathcal{A})$ is admissible with respect to \mathcal{A} (or \mathcal{A} -admissible), if $\mathfrak{S}_{\text{prin}}(\mathcal{A}) \subset \mathfrak{S} \subset \mathfrak{S}_{\text{cyc}}(\mathcal{A})$ and \mathfrak{S} is closed under taking \mathcal{A} -quasisubsections.*

Given an \mathcal{A} -admissible class \mathfrak{S} and an \mathcal{A} -section S we set $\mathfrak{S}_S = \rho_{G,S}(\mathfrak{S})$ where the mapping $\rho_{G,S}$ is defined in Subsection 4.1. Since $\rho_{G,S}$ is identical on the set $\mathfrak{S}(S) \subset \mathfrak{S}(G)$, we have

$$(41) \quad \mathfrak{S}_S = \{\rho_{G,S}(S') : S' \in \mathfrak{S}, S' \preceq S\}.$$

The following statement gives the properties of admissible classes.

Lemma 10.2. *Let \mathfrak{S} be an admissible class with respect to a quasidense circulant S -ring \mathcal{A} . Then*

- (1) $\mathfrak{S}_0(\mathcal{A}) \subset \mathfrak{S}$,
- (2) *the class \mathfrak{S}_S is \mathcal{A}_S -admissible for any $S \in \mathfrak{S}(\mathcal{A})$,*
- (3) *if $S \in \mathfrak{S}(\mathcal{A}) \setminus \mathfrak{S}$, then \mathcal{A}_S admits a gwr-decomposition.*

Proof. Statements (1) is obvious. Statement (2) follows from (41). To prove Statement (3) let $S \in \mathfrak{S}(\mathcal{A}) \setminus \mathfrak{S}$. Then $\text{rad}(\mathcal{A}_S) \neq 1$, for otherwise S is a subsection of a principal \mathcal{A} -section (Lemma 5.3) whereas \mathfrak{S} contains all such sections. Thus the required statement follows from Theorem 5.2. ■

Let \mathcal{A} be a quasidense circulant S -ring and \mathfrak{S} an \mathcal{A} -admissible class. Then there is at least one coset S -ring \mathcal{A}' , namely the group ring, for which relations (40) hold for all $S \in \mathfrak{S}$. This justifies the following definition.

Definition 10.3. *The coset closure $\mathcal{A}_{0,\mathfrak{S}}$ of \mathcal{A} with respect to \mathfrak{S} is the intersection of all coset S -rings \mathcal{A}' such that relations (40) hold for all $S \in \mathfrak{S}$.*

Clearly, $\mathcal{A}_{0,\mathfrak{S}} \geq \mathcal{A}$. Moreover, from (6) it follows that $(\mathcal{A}_{0,\mathfrak{S}})_S = \mathbb{Z}S$ for all $S \in \mathfrak{S}$. Besides, the discussion in the first paragraph of the section shows that

$$(42) \quad \mathcal{A}_0 = \mathcal{A}_{0,\mathfrak{S}} \quad \text{when} \quad \mathfrak{S} = \mathfrak{S}_0(\mathcal{A}).$$

The sense of the following definition will be clarified in Corollary 10.10. With any \mathcal{A} -admissible class \mathfrak{S} we associate a larger class of sections defined as follows

$$\widehat{\mathfrak{S}} = \{S \in \mathfrak{S}(\mathcal{A}) : S_p \in \mathfrak{S} \text{ for all primes } p \text{ dividing } |S|\}.$$

The class $\widehat{\mathfrak{S}}$ is closed under taking quasisubsections, but generally is not admissible because may contain non-cyclotomic sections. As the example on page 21 shows, in general it can be larger than \mathfrak{S} .

Lemma 10.4. *Let \mathcal{A} be a quasidense S -ring over a cyclic group G . Then for any \mathcal{A} -admissible class \mathfrak{S} we have*

- (1) *the S -ring $\mathcal{A}_{0,\mathfrak{S}}$ is a coset one,*
- (2) *$(\mathcal{A}_{0,\mathfrak{S}})_S = \mathbb{Z}S$ for all $S \in \mathfrak{S}$,*
- (3) *any coset belonging to a section in $\widehat{\mathfrak{S}}$ is an $\mathcal{A}_{0,\mathfrak{S}}$ -set.*

Proof. Statement (1) immediately follows from Theorem 8.5. Statement (2) follows from the remark after the definition of the coset closure $\mathcal{A}_{0,\mathfrak{S}}$. To prove statement (3) let \mathcal{A}' be a coset S -ring such that relations (40) hold for all $S \in \mathfrak{S}$.

Then given a section $S \in \widehat{\mathfrak{S}}$ we have $S_p \in \mathfrak{S}$, and hence $(\mathcal{A}')_{S_p} = \mathbb{Z}S_p$ for all primes p dividing $|S|$. This implies that $(\mathcal{A}')_S = \mathbb{Z}S$. It follows that if $S = U/L$, then any L -coset in U is an \mathcal{A}' -set. Since \mathcal{A} is the intersection of all such \mathcal{A}' , it is an \mathcal{A} -set as required. ■

10.2. Lifting. Let \mathfrak{S} be an admissible class with respect to a quasidense S-ring \mathcal{A} over a cyclic group G . Suppose that \mathcal{A} is the S -wreath product where $S = U/L$ is an \mathcal{A} -section. We say that this product is \mathfrak{S} -consistent if any section in \mathfrak{S} is either an \mathcal{A}_U - or $\mathcal{A}_{G/L}$ -section. Below to simplify notation we write $(\mathcal{A}_S)_{0,\mathfrak{S}}$ instead of $(\mathcal{A}_S)_{0,\mathfrak{S}_S}$.

Theorem 10.5. *Let \mathcal{A} be a circulant quasidense S-ring, \mathfrak{S} an \mathcal{A} -admissible class and S an \mathcal{A} -section. Then the following conditions are equivalent:*

- (1) $(\mathcal{A}_S)_{0,\mathfrak{S}} \neq \mathbb{Z}S$,
- (2) $S \notin \widehat{\mathfrak{S}}$,
- (3) *there exists an \mathfrak{S}_S -consistent gwr-decomposition of \mathcal{A}_S ,*
- (4) *there exists an \mathfrak{S}_S -consistent gwr-decomposition of \mathcal{A}_S that can be lifted to an \mathfrak{S} -consistent gwr-decomposition of \mathcal{A} .*

Proof. Let us prove implication (1) \Rightarrow (2). Suppose on the contrary that $S \in \widehat{\mathfrak{S}}$. Then obviously $S \in \widehat{\mathfrak{S}_S}$. By statement (3) of Lemma 10.4 applied to $\mathcal{A} = \mathcal{A}_S$, $\mathfrak{S} = \mathfrak{S}_S$ and the section $S/1$, this implies that $(\mathcal{A}_S)_{0,\mathfrak{S}} = \mathbb{Z}S$. Contradiction.

To prove implication (2) \Rightarrow (4) let $S \notin \widehat{\mathfrak{S}}$. Then there is a prime divisor p of $|S|$ such that $S_p \notin \mathfrak{S}$. Suppose first that S_p is not an \mathcal{A}_S -group. Then the S-rings \mathcal{A}_S and \mathcal{A} are not dense. Then by Theorem 6.2 there exists an elementary nondense \mathcal{A}_S -section S_0 . By the first part of Theorem 6.5 applied to the S-rings \mathcal{A} and \mathcal{A}_S , and equality (20) there are gwr-decompositions

$$(43) \quad \mathcal{A} = \mathcal{A}_U \wr_{U/L} \mathcal{A}_{G/L} \quad \text{and} \quad \mathcal{A}_S = \mathcal{A}_{U_S} \wr_{U_S/L_S} \mathcal{A}_{S/L_S}$$

where $U = U(S_0)$ and $L = L(S_0)$ and $U_S/L_S = \rho_{G,S}(U/L)$. It follows that the first one is a lifting of the second. Finally, the second part of Theorem 6.5 together with the fact that any section in the class \mathfrak{S} is dense, shows that these gwr-decompositions are \mathfrak{S} - and \mathfrak{S}_S -consistent respectively. Thus statement (4) holds in this case.

Let now S_p be an \mathcal{A}_S -group. Then by Lemma 10.2 the hypothesis of Theorem 7.3 holds for the S-rings \mathcal{A} and \mathcal{A}_S with $S_0 = S_p$ in both cases. So by statement (1) of this theorem there are gwr-decompositions (43) and due to (25) the first one is a lifting of the second. To prove that the first decomposition is \mathfrak{S} -consistent, suppose on the contrary that there exists $T \in \mathfrak{S}$ which is neither \mathcal{A}_U - nor $\mathcal{A}_{G/L}$ -section. Then by statement (2) of that theorem S_p is a quasisubsection of T . Therefore $S_p \in \mathfrak{S}$, which contradicts the assumption on S_p . The \mathfrak{S} -consistency of the second decomposition is proved similarly. Thus statement (4) holds in this case too.

Implication (4) \Rightarrow (3) is obvious. To prove implication (3) \Rightarrow (1) without loss of generality we can assume that $S = G$. Suppose that the S-ring \mathcal{A} admits an \mathfrak{S} -consistent U/L -decomposition. By Theorem 2.1 we can form the S-ring

$$\mathcal{B} = \mathcal{A}'_U \wr_{U/L} \mathcal{A}'_{G/L}$$

where $\mathcal{A}' = \mathcal{A}_{0,\mathfrak{S}}$. By Theorem 8.4 this S-ring is a coset one. Moreover, the consistency property implies that any section $T \in \mathfrak{S}$ is either an \mathcal{A}_U - or $\mathcal{A}_{G/L}$ -section, and hence either an \mathcal{A}'_U - or $\mathcal{A}'_{G/L}$ -section. Therefore $\mathcal{B}_T = \mathbb{Z}T$ for all such T . Thus, by the definition of the coset closure we have $\mathcal{B} \geq \mathcal{A}'$. So from the minimality property of the generalized wreath product it follows that the S-ring \mathcal{A}' admits the U/L -decomposition. Therefore $\mathcal{A}' \neq \mathbb{Z}G$ as required. ■

The following auxiliary statement will be used in proving the theorems in the next subsection.

Lemma 10.6. *Let \mathcal{A} be a circulant quasidense S-ring, \mathfrak{S} an \mathcal{A} -admissible class and S an \mathcal{A} -section. Suppose that \mathcal{A}_S admits an \mathfrak{S}_S -consistent T -decomposition that can be lifted to an \mathfrak{S} -consistent gwr-decomposition of \mathcal{A} . Then $(\mathcal{A}_{0,\mathfrak{S}})_S$ admits the T -decomposition.*

Proof. By Theorem 4.3 without loss of generality we can assume that $S = G$. Then we have to verify $\mathcal{A}_{0,\mathfrak{S}}$ admits the T -decomposition whenever \mathcal{A} admits an \mathfrak{S} -consistent T -decomposition. However, the latter implies that any $T' \in \mathfrak{S}$ is either an \mathcal{A}_U - or $\mathcal{A}_{G/L}$ -section where $U/L = T$. It follows that T' is either \mathcal{A}'_U - or $\mathcal{A}'_{G/L}$ -section where \mathcal{A}' is the elementary coset S-ring $\mathbb{Z}(G, T)$. Therefore

$$(\mathcal{A}')_{T'} = \mathbb{Z}T', \quad T' \in \mathfrak{S}.$$

So by the definition of the coset closure we have $\mathcal{A}' \geq \mathcal{A}_{0,\mathfrak{S}}$. Since obviously T is an $\mathcal{A}_{0,\mathfrak{S}}$ -section, the S-ring $\mathcal{A}_{0,\mathfrak{S}}$ is the T -wreath product, as required. ■

10.3. Main properties. Here we study the coset closure in detail and, in particular, find its explicit structure.

Theorem 10.7. *Let \mathcal{A} be a quasidense circulant S-ring. Then $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{A}_{0,\mathfrak{S}})$ for any \mathcal{A} -admissible class \mathfrak{S} .*

Proof. Below to simplify notations we omit the letter \mathfrak{S} in subscript. The theorem statement is obviously true when the S-ring \mathcal{A} is dense. Suppose that it is not dense. Then by Theorems 6.2 and 6.5 the S-ring \mathcal{A} admits an \mathfrak{S} -consistent U/L -decomposition.

Lemma 10.8. *Let \mathcal{B} be a circulant S-ring over G . Suppose that \mathcal{B} is a U/L -wreath product. Then*

$$\mathcal{G}(\mathcal{B}) = \mathcal{G}(\mathcal{B}_U) \cup \pi^{-1}(\mathcal{G}(\mathcal{B}_{G/L}))$$

where $\pi = \pi_{G/L}$ is the quotient epimorphism from G to G/L .

Proof. Obviously, the right-hand side is contained in the left-hand one. Conversely, let $H \in \mathcal{G}(\mathcal{B})$. Without loss of generality we can assume that $H \not\leq U$. Then any highest basic set of \mathcal{B}_H is outside U . Since \mathcal{B} is a U/L -wreath product, the radical of that set contains L . Thus $L \leq H$ as required. ■

By Lemma 10.8 and Lemma 10.6 with $S = G$ this implies that

$$\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{A}_U) \cup \pi^{-1}(\mathcal{G}(\mathcal{A}_{G/L})) \quad \text{and} \quad \mathcal{G}(\mathcal{A}_0) = \mathcal{G}((\mathcal{A}_0)_U) \cup \pi^{-1}(\mathcal{G}((\mathcal{A}_0)_{G/L})).$$

On the other hand, by induction we have

$$(44) \quad \mathcal{G}(\mathcal{A}_U) = \mathcal{G}((\mathcal{A}_U)_0) \quad \text{and} \quad \mathcal{G}(\mathcal{A}_{G/L}) = \mathcal{G}((\mathcal{A}_{G/L})_0).$$

Thus it suffices to verify that given $S \in \{U, G/L\}$ we have

$$(45) \quad \mathcal{G}((\mathcal{A}_S)_0) = \mathcal{G}((\mathcal{A}_0)_S).$$

By Lemma 6.6 with $\mathcal{A} = (\mathcal{A}_S)_0$ and $\mathcal{A}' = (\mathcal{A}_0)_S$ all we need to prove is that any elementary nondense section of $(\mathcal{A}_S)_0$ is an elementary nondense section of $(\mathcal{A}_0)_S$. However by (44), any such section T is an \mathcal{A}_S -section. Therefore T is an elementary nondense section of \mathcal{A} . In particular, \mathcal{A}_T admits a unique \mathfrak{S} -consistent gwr-decomposition, and by Theorem 6.5 this decomposition can be lifted to an \mathfrak{S} -consistent gwr-decomposition of \mathcal{A} . By Lemma 10.6 this implies that T is an elementary nondense section of \mathcal{A}_0 , and hence of $(\mathcal{A}_0)_S$. ■

It is easily seen that $(\mathcal{A}_S)_{0,\mathfrak{S}} \leq (\mathcal{A}_{0,\mathfrak{S}})_S$ for all \mathcal{A} -sections S . The following theorem refines this simple statement.

Theorem 10.9. *Let \mathcal{A} be a quasidense circulant S-ring and \mathfrak{S} an \mathcal{A} -admissible class. Then $(\mathcal{A}_S)_{0,\mathfrak{S}} = (\mathcal{A}_{0,\mathfrak{S}})_S$ for any \mathcal{A} -section S .*

Proof. Below to simplify notations we omit the letter \mathfrak{S} in subscript. Suppose on the contrary that $(\mathcal{A}_S)_0 < (\mathcal{A}_0)_S$ for some \mathcal{A} -section S . Then there exist basic sets X and Y of the S-rings $(\mathcal{A}_S)_0$ and $(\mathcal{A}_0)_S$ respectively, such that Y is a proper subset of X . From Theorem 10.7 it follows that

$$\mathcal{G}((\mathcal{A}_0)_S) = \mathcal{G}(\mathcal{A}_S) = \mathcal{G}((\mathcal{A}_S)_0).$$

Therefore $\langle Y \rangle$ is an $(\mathcal{A}_S)_0$ -group. However, the set $X \cap \langle Y \rangle$ is not empty. Thus, $X \subset \langle Y \rangle$. On the other hand, X and Y are cosets, because the S-rings $(\mathcal{A}_S)_0$ and $(\mathcal{A}_0)_S$ are coset ones. Since Y is a proper subset of X , this implies that $\text{rad}(Y) < \text{rad}(X)$. Thus, $((\mathcal{A}_S)_0)_T \neq \mathbb{Z}T$ where $T = \langle Y \rangle / \text{rad}(Y)$. This implies that

$$(\mathcal{A}_T)_0 = ((\mathcal{A}_S)_T)_0 \leq ((\mathcal{A}_S)_0)_T < \mathbb{Z}T.$$

By implication (1) \Rightarrow (4) of Theorem 10.5 the hypothesis of Lemma 10.6 is satisfied for $S = T$. Therefore the S-ring $(\mathcal{A}_0)_T$ admits a gwr-decomposition. It follows that $\text{rad}((\mathcal{A}_0)_T) \neq 1$, which is impossible because T is a principal section of the S-ring $(\mathcal{A}_0)_S$. ■

Corollary 10.10. *In the conditions of Theorem 10.9 the following statements hold:*

- (1) $\widehat{\mathfrak{S}} = \mathfrak{S}_0(\mathcal{A}_{0,\mathfrak{S}})$,
- (2) $\widehat{\mathfrak{S}}$ is an \mathcal{A} -admissible class if and only if $\widehat{\mathfrak{S}} \subset \mathfrak{S}_{cyc}(\mathcal{A})$,
- (3) $\mathfrak{S}_0 = \mathfrak{S}_0(\mathcal{A}_0) = \widehat{\mathfrak{S}_0(\mathcal{A})}$ where \mathfrak{S}_0 and \mathcal{A}_0 are as in Introduction.

Proof. Statement (1) immediately follows from equivalence (1) \Leftrightarrow (2) of Theorem 10.5 and Theorems 10.9 and 8.3. Statement (2) follows from statement (1) and Theorem 10.7. Statement (3) is a special case of statement (1) for $\mathfrak{S} = \mathfrak{S}_0(\mathcal{A})$. ■

Let \mathcal{A} be a quasidense circulant S-ring and \mathfrak{S} an \mathcal{A} -admissible class. For a basic set X of \mathcal{A} set

$$(46) \quad L_{\mathfrak{S}}(X) = \bigcap_{\langle X \rangle / L \in \widehat{\mathfrak{S}}, L \leq \text{rad}(X)} L.$$

Certainly, at least one group L does exist because $\widehat{\mathfrak{S}} \supset \mathfrak{S}_{prin}(\mathcal{A})$, and hence one can take $L = \text{rad}(X)$. It should be mentioned that since the class $\widehat{\mathfrak{S}}$ is closed with respect to taking subsections, the left-hand side of (46) does not change when the intersection is taken over all sections $U/L \in \widehat{\mathfrak{S}}$ having $\langle X \rangle / \text{rad}(X)$ as a subsection. Clearly, $L_{\mathfrak{S}}(X)$ is an \mathcal{A} -group contained in $\text{rad}(X)$. Therefore X is a union of cosets of it; the set of all of them is denoted by $X/L_{\mathfrak{S}}(X)$.

Theorem 10.11. *Let \mathcal{A} be a quasidense circulant S-ring and \mathfrak{S} an \mathcal{A} -admissible class. Then*

- (1) $\mathcal{S}(\mathcal{A}_{0,\mathfrak{S}}) = \bigcup_{S \in \mathcal{S}(\mathcal{A})} X/L_{\mathfrak{S}}(X) = \{xL_{\mathfrak{S}}(X) : x \in X \in \mathcal{S}(\mathcal{A})\}$,
- (2) *elements x and y of a basic set X of \mathcal{A} are in the same basic set of $\mathcal{A}_{0,\mathfrak{S}}$ if and only if $\pi_S(x) = \pi_S(y)$ for any section $S \in \widehat{\mathfrak{S}}$ having $\langle X \rangle / \text{rad}(X)$ as a subsection.*

Proof. Statement (2) immediately follows from statement (1) and the remark before the theorem. To prove statement (1) let $X \in \mathcal{S}(\mathcal{A})$ and $x \in X$. Denote by X_0 the basic set of the S-ring $\mathcal{A}_{0,\mathfrak{S}}$ that contains x . Then $X_0 = xL_0$ for some $\mathcal{A}_{0,\mathfrak{S}}$ -subgroup L_0 , because $\mathcal{A}_{0,\mathfrak{S}}$ is a coset S-ring. Moreover, by statement (3) of Lemma 10.4 the set X_0 is contained in some L -coset in $U := \langle X \rangle$ for any group L such that $U/L \in \widehat{\mathfrak{S}}$. By (46) this implies that

$$L_0 \leq L_{\mathfrak{S}}(X).$$

If $L_0 = L_{\mathfrak{S}}(X)$, then $xL_{\mathfrak{S}}(X) = X_0$ and we are done. Suppose that $L_0 < L_{\mathfrak{S}}(X)$. By Theorem 10.7 the group $U_0 := \langle X_0 \rangle$ is an \mathcal{A} -group. Since X intersects U_0 , this implies that $\langle X \rangle = U_0$. Moreover, due to the assumption we also have $L_0 < \text{rad}(X)$. On the other hand, since $S := U_0/L_0$ is a principal section of a coset S-ring $\mathcal{A}_{0,\mathfrak{S}}$, Theorem 8.3 implies that $(\mathcal{A}_{0,\mathfrak{S}})_S = \mathbb{Z}S$. By Theorem 10.9 and implication (2) \Rightarrow (1) of Theorem 10.5 this implies that $S \in \widehat{\mathfrak{S}}$. Since $\langle X \rangle / \text{rad}(X)$ is a subsection of S , we obtain that $L_{\mathfrak{S}}(X) \leq L_0$. Contradiction. ■

11. MULTIPLIERS

Let \mathcal{A} be a quasidense circulant S-ring and \mathfrak{S} an \mathcal{A} -admissible class. In what follows for an \mathcal{A} -section S we set $\text{Aut}_{\mathcal{A}}(S) = \text{Aut}(\mathcal{A}_S) \cap \text{Aut}(S)$.

Definition 11.1. *An element $\Sigma = \{\sigma_S\}$ of the direct product $\prod_{S \in \mathfrak{S}} \text{Aut}_{\mathcal{A}}(S)$, is called an \mathfrak{S} -multiplier of \mathcal{A} if the following two conditions are satisfied for all sections $S_1, S_2 \in \mathfrak{S}$:*

- (M1) *if $S_1 \succeq S_2$, then $(\sigma_{S_1})^{S_2} = \sigma_{S_2}$,*
- (M2) *if $S_1 \sim S_2$ implies $m(\sigma_{S_1}) = m(\sigma_{S_2})$.*

The group of all \mathfrak{S} -multipliers of \mathcal{A} is denoted by $\text{Mult}_{\mathfrak{S}}(\mathcal{A})$.

It should be noted that if the class \mathfrak{S}_0 defined in (3) is contained in $\mathfrak{S}_{\text{cyc}}(\mathcal{A})$, then it is admissible (Corollary 10.10) and $\text{Mult}(\mathcal{A}) = \text{Mult}_{\mathfrak{S}_0}(\mathcal{A})$. The following lemma gives a natural way to construct \mathfrak{S} -multipliers.

Lemma 11.2. *Suppose that we are given $\gamma \in \text{Aut}(\mathcal{A})$ such that $\gamma^S \in \text{Aut}_{\mathcal{A}}(S)$ for all $S \in \mathfrak{S}$. Then the family $\Sigma(\gamma) = \{\sigma_S(\gamma)\}_{S \in \mathfrak{S}}$ where $\sigma_S(\gamma) = \gamma^S$, is an \mathfrak{S} -multiplier.*

Proof. Condition (M1) is obvious. Besides, from Theorem 4.2 it follows that $m(\gamma^{S_1}) = m(\gamma^{S_2})$ for any projectively equivalent sections $S_1, S_2 \in \mathfrak{S}$. Therefore $m(\sigma_{S_1}(\gamma)) = m(\sigma_{S_2}(\gamma))$. Thus condition (M2) is also satisfied for $\Sigma(\gamma)$, and we are done. ■

Let $\Sigma \in \text{Mult}_{\mathfrak{S}}(\mathcal{A})$ and $T \in \mathfrak{S}(\mathcal{A})$. Then \mathfrak{S}_T is an \mathcal{A}_T -admissible class by statement (2) of Lemma 10.2, and $\mathfrak{S}_T \subset \mathfrak{S}$ by (41). Conditions (M1) and (M2) are obviously satisfied for the restriction Σ^T of Σ to \mathfrak{S}_T , which is by definition

considered as an element of the direct product $\prod_{S \in \mathfrak{S}_T} \text{Aut}_{\mathcal{A}}(S)$. This proves the following statement.

Lemma 11.3. *The family Σ^T is an \mathfrak{S}_T -multiplier of the S-ring \mathcal{A}_T . ■*

We are going to construct \mathfrak{S} -multipliers of \mathcal{A} by means of similarities belonging to the set

$$\Phi_{0,\mathfrak{S}}(\mathcal{A}) = \{\varphi \in \Phi(\mathcal{A}_{0,\mathfrak{S}}) : X^\varphi = X \text{ for all } X \in \mathcal{S}(\mathcal{A})\}.$$

Namely, with any $\varphi \in \Phi_{0,\mathfrak{S}}(\mathcal{A})$ we associate an element $\Sigma_\varphi = \{\sigma_S\}$ of the group $\prod_{S \in \mathfrak{S}} \text{Aut}_{\mathcal{A}}(S)$ where the automorphisms σ_S are defined as follows. Let $S \in \mathfrak{S}$. Then by statement (2) of Lemma 10.4 we have $(\mathcal{A}_{0,\mathfrak{S}})_S = \mathbb{Z}S$. Therefore there exists a uniquely determined automorphism $\sigma_S \in \text{Aut}(S)$ that induces the restriction φ_S of the similarity φ to S (Lemma 3.2). From the choice of φ it follows that $\sigma_S \in \text{Aut}_{\mathcal{A}}(S)$.

Lemma 11.4. *The family Σ_φ is an \mathfrak{S} -multiplier of the S-ring \mathcal{A} . Moreover, $\varphi_T \in \Phi_{0,\mathfrak{S}_T}(\mathcal{A}_T)$ and $(\Sigma_\varphi)^T = \Sigma_{\varphi_T}$ for all $T \in \mathfrak{S}(\mathcal{A})$.*

Proof. By Theorem 9.1 the similarity φ is induced by an isomorphism γ of the S-ring $\mathcal{A}_{0,\mathfrak{S}}$. Clearly, γ can be chosen so that $1^\gamma = 1$. Then by Lemma 3.2 we have $\gamma^S \in \text{Aut}_{\mathcal{A}}(S)$ for all $S \in \mathfrak{S}$. Besides, $\gamma \in \text{Aut}(\mathcal{A})$ by Lemma 3.7. So from Lemma 11.2 it follows that $\Sigma(\gamma)$ is an \mathfrak{S} -multiplier of \mathcal{A} . Thus, the first statement follows because $\Sigma_\varphi = \Sigma(\gamma)$.

To prove the second statement, let $T \in \mathfrak{S}(\mathcal{A})$. Then $\varphi_T \in \Phi((\mathcal{A}_{0,\mathfrak{S}})_T)$. By Theorem 10.9 this implies that $\varphi_T \in \Phi((\mathcal{A}_T)_{0,\mathfrak{S}})$. So $\varphi_T \in \Phi_{0,\mathfrak{S}_T}(\mathcal{A}_T)$. The rest of the statement easily follows from the definitions. ■

To simplify notation we will write $\Phi_{0,\mathfrak{S}}(\mathcal{A}_T)$ instead of $\Phi_{0,\mathfrak{S}_T}(\mathcal{A}_T)$.

Theorem 11.5. *Let \mathcal{A} be an quasidense circulant S-ring. Then the mapping*

$$(47) \quad \Phi_{0,\mathfrak{S}}(\mathcal{A}) \rightarrow \text{Mult}_{\mathfrak{S}}(\mathcal{A}), \quad \varphi \mapsto \Sigma_\varphi$$

is a group isomorphism for any \mathcal{A} -admissible class \mathfrak{S} .

Proof. The mapping (47) is obviously a group homomorphism. To prove its injectivity suppose that $\Sigma_\varphi = \Sigma_\psi$ for some $\varphi, \psi \in \Phi_{0,\mathfrak{S}}(\mathcal{A})$. Then obviously $\varphi_S = \psi_S$ for all $S \in \mathfrak{S}$. By Lemma 3.2 the equality also holds for all $S \in \widehat{\mathfrak{S}}$ because any similarity of $\mathbb{Z}S$ is uniquely determined by its restrictions to the Sylow sections S_p . However, $\widehat{\mathfrak{S}} = \mathfrak{S}_0(\mathcal{A}_{0,\mathfrak{S}})$ by Corollary 10.10. Therefore φ and ψ are equal on all principal $\mathcal{A}_{0,\mathfrak{S}}$ -sections. Thus $\varphi = \psi$ by Lemma 3.1.

Let us prove the surjectivity of homomorphism (47) by induction on the size of the group G underlying \mathcal{A} . Let $\Sigma = \{\sigma_S\}$ be an \mathfrak{S} -multiplier of \mathcal{A} . First, suppose that $\mathcal{A}_{0,\mathfrak{S}} = \mathbb{Z}G$. By implication (1) \Rightarrow (2) of Theorem 10.5 this implies that $G \in \widehat{\mathfrak{S}}$. So any Sylow subgroup of G belongs to \mathfrak{S} . It follows that \mathcal{A} is a subtensor product of cyclotomic S-rings. In this case the following statement holds.

Lemma 11.6. *There exists $\gamma \in \text{Aut}_{\mathcal{A}}(G)$ such that $\Sigma = \Sigma(\gamma)$.*

Proof. Set γ to be the unique automorphism of G such that $\gamma^{G_p} = \sigma_{G_p}$ for all Sylow subgroups G_p of G . We claim that

$$(48) \quad \sigma_S(\gamma) = \sigma_S, \quad S \in \mathfrak{S},$$

where $\sigma_S(\gamma)$ is the S -component of the family $\Sigma(\gamma)$. Indeed, the automorphism σ_S is uniquely determined by its p -components $(\sigma_S)_p \in \text{Aut}(S_p)$. Besides, $S_p \in \mathfrak{S}$ because $S \in \mathfrak{S}$, and hence $(\sigma_S)_p = \sigma_{S_p}$ by Definition 11.1. Since $G_p \in \mathfrak{S}$, each σ_{S_p} is in its turn uniquely determined by the automorphism σ_{G_p} . By the definition of γ this proves (48).

To complete the proof of the lemma let us verify that $\gamma \in \text{Aut}(\mathcal{A})$. Since $\gamma \in \text{Aut}(G)$, by Lemma 3.7 for $\mathcal{A}' = \mathbb{Z}G$, it suffices to check that $X^\gamma = X$ for all basic sets X of \mathcal{A} . However, the section $S = \langle X \rangle / \text{rad}(X)$ belongs to \mathfrak{S} . Moreover, X is a disjoint union of $\text{rad}(X)$ -cosets. Since $\sigma_S \in \text{Aut}(\mathcal{A}_S)$, equality (48) implies that the automorphism γ permutes the cosets in this union, as required. ■

By Lemma 11.6 we have $\Sigma = \Sigma(\gamma)$. On the other hand, it is easily seen that $\Sigma(\gamma) = \Sigma_\varphi$ where $\varphi = \varphi_\gamma$ is the similarity induced by the automorphism γ (Lemma 3.2). Thus $\Sigma = \Sigma_\varphi$ as required.

Now, assume that $\mathcal{A}_{0,\mathfrak{S}} \neq \mathbb{Z}G$. Then by implication (1) \Rightarrow (3) of Theorem 10.5 for $S = G$, there exists an \mathfrak{S} -consistent U/L -decomposition of \mathcal{A} . By the inductive hypothesis applied to the S -ring \mathcal{A}_U and \mathfrak{S}_U -multiplier Σ^U , as well to the S -ring $\mathcal{A}_{G/L}$ and $\mathfrak{S}^{G/L}$ -multiplier $\Sigma^{G/L}$, there exist similarities $\varphi_1 \in \Phi_{0,\mathfrak{S}}(\mathcal{A}_U)$ and $\varphi_2 \in \Phi_{0,\mathfrak{S}}(\mathcal{A}_{G/L})$ such that

$$(49) \quad \Sigma^U = \Sigma_{\varphi_1} \quad \text{and} \quad \Sigma^{G/L} = \Sigma_{\varphi_2}.$$

Furthermore, $(\mathcal{A}_U)_{0,\mathfrak{S}} = (\mathcal{A}_{0,\mathfrak{S}})_U$ and $(\mathcal{A}_{G/L})_{0,\mathfrak{S}} = (\mathcal{A}_{0,\mathfrak{S}})_{G/L}$ by Theorem 10.9. Therefore

$$(50) \quad ((\mathcal{A}_U)_{0,\mathfrak{S}})_{U/L} = (\mathcal{A}_{0,\mathfrak{S}})_{U/L} = ((\mathcal{A}_{G/L})_{0,\mathfrak{S}})_{U/L}.$$

However, again by Theorem 10.9 we have $(\mathcal{A}_{0,\mathfrak{S}})_{U/L} = (\mathcal{A}_{U/L})_{0,\mathfrak{S}}$. It follows that the similarities $(\varphi_1)^{U/L}$ and $(\varphi_2)^{U/L}$ belong to $\Phi_{0,\mathfrak{S}}(\mathcal{A}_{U/L})$.

By the first equality in (49) and Lemma 11.4 applied to $\mathcal{A} = \mathcal{A}_U$, $\varphi = \varphi_1$ and $T = U/L$, we obtain that

$$\Sigma_{(\varphi_1)^{U/L}} = (\Sigma_{\varphi_1})^{U/L} = (\Sigma^U)^{U/L} = \Sigma^{U/L}.$$

Similarly, by the second equality in (49) and Lemma 11.4 applied to $\mathcal{A} = \mathcal{A}_U$, $\varphi = \varphi_1$ and $T = U/L$ we have

$$\Sigma_{(\varphi_2)^{U/L}} = (\Sigma_{\varphi_2})^{U/L} = (\Sigma^{G/L})^{U/L} = \Sigma^{U/L}.$$

Thus $\Sigma_{(\varphi_1)^{U/L}} = \Sigma_{(\varphi_2)^{U/L}}$ and by the injectivity statement we have

$$(\varphi_1)^{U/L} = (\varphi_2)^{U/L}.$$

On the other hand, by Theorem 2.1 and equality (50) one can form the U/L -wreath product \mathcal{A}' of the S -rings $(\mathcal{A}_U)_{0,\mathfrak{S}}$ and $(\mathcal{A}_{G/L})_{0,\mathfrak{S}}$. Thus by statement (2) of Theorem 3.3 there exists a uniquely determined similarity $\varphi \in \Phi(\mathcal{A}')$ such that

$$(51) \quad \varphi^U = \varphi_1 \quad \text{and} \quad \varphi^{G/L} = \varphi_2.$$

However, \mathcal{A} is the U/L -wreath product. By Corollary 10.6 so is $\mathcal{A}_{0,\mathfrak{S}}$. Since the restrictions of the latter S -ring to U and G/L coincide with the corresponding restrictions of the S -ring \mathcal{A}' (see above), we conclude that $\mathcal{A}' = \mathcal{A}_{0,\mathfrak{S}}$. Thus $\varphi \in \Phi(\mathcal{A}_{0,\mathfrak{S}})$. Since $\varphi_1 \in \Phi_{0,\mathfrak{S}}(\mathcal{A}_U)$ and $\varphi_2 \in \Phi_{0,\mathfrak{S}}(\mathcal{A}_{G/L})$, we conclude that $\varphi \in \Phi_{0,\mathfrak{S}}(\mathcal{A})$.

To complete the proof let us verify that $\Sigma = \Sigma_\varphi$. To do this we observe that from (49) and (51) it follows that $(\Sigma_\varphi)^U = \Sigma^U$ and $(\Sigma_\varphi)^{G/L} = \Sigma^{G/L}$. This proves the required statement because by the \mathfrak{S} -consistency property, any section in \mathfrak{S} is either \mathcal{A}_U - or $\mathcal{A}_{G/L}$ -section. ■

12. PROOF OF THEOREMS 1.2 AND 1.3

The following auxiliary statement is interesting by itself. In particular, it shows that the subgroup lattices of a quasidense S-ring and its schurian closure are equal. It seems that this statement could be generalized to all circulant S-rings.

Lemma 12.1. *Let \mathcal{A} be a quasidense circulant S-ring and \mathcal{A}' its schurian closure. Then $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{A}')$ and $\mathcal{A}_0 = (\mathcal{A}')_0$.*

Proof. From Theorem 9.1 it follows that any coset S-ring that contains \mathcal{A} , contains also \mathcal{A}' . This proves the second equality and shows that $\mathcal{A}_0 \geq \mathcal{A}'$. Since obviously $\mathcal{A}' \geq \mathcal{A}$, the first equality follows from equality (42) and Theorem 10.7. ■

Proof of Theorem 1.2. Without loss of generality we can assume that the S-ring \mathcal{A} is schurian. Indeed, from Lemma 12.1 it follows that $\mathcal{A}_0 = (\mathcal{A}')_0$ where $\mathcal{A}' = \text{Sch}(\mathcal{A})$. Therefore it suffices to verify that

$$(52) \quad \Phi_0(\mathcal{A}) = \Phi_0(\mathcal{A}').$$

However, by Theorem 9.1 any similarity $\varphi \in \Phi_0(\mathcal{A})$ is induced by an isomorphism f of \mathcal{A}_0 to itself. Without loss of generality we can assume that $1^f = 1$. Then by (11) and the choice of φ we have $X^f = X^\varphi = X$ for all $X \in \mathcal{S}(\mathcal{A})$. By Lemma 3.7 this implies that $f \in \text{Aut}(\mathcal{A})$, and hence $f \in \text{Aut}(\mathcal{A}')$. Then φ is identical on \mathcal{A}' , and so $\varphi \in \Phi_0(\mathcal{A}')$. Thus $\Phi_0(\mathcal{A}) \subset \Phi_0(\mathcal{A}')$. Since the reverse inclusion is obvious, equality (52) follows.

From the definition of the group $\Phi_0 = \Phi_0(\mathcal{A})$ it follows that $\mathcal{A} \leq \mathcal{B}$ where $\mathcal{B} = (\mathcal{A}_0)^{\Phi_0}$. To prove the reverse inclusion we have to verify that any basic set of \mathcal{A} is contained in a basic set of \mathcal{B} . Let x and y belong to the same basic set of \mathcal{A} . Then it suffices to verify that

$$(53) \quad x^\delta = y \quad \text{for some } \delta \in \text{Aut}(\mathcal{B}), \quad 1^\delta = 1.$$

To do this we recall that \mathcal{A} is schurian. So by Theorem 5.8 one can find a group $\Gamma \in \mathcal{M}(\mathcal{A})$ such that

$$(54) \quad \Gamma^S = \text{Hol}_{\mathcal{A}}(S), \quad S \in \mathfrak{S}_0(\mathcal{A}).$$

It follows that there exists $\gamma \in \Gamma$ such that $1^\gamma = 1$ and $x^\gamma = y$. Due to (54) we have $\gamma^S \in \text{Aut}_{\mathcal{A}}(S)$ for all $S \in \mathfrak{S}_0(\mathcal{A})$. By Lemma 11.2 this implies that the family $\Sigma = \{\gamma^S\}$ is an $\mathfrak{S}_0(\mathcal{A})$ -multiplier of \mathcal{A} . Therefore by Theorem 11.5 there exists a uniquely determined similarity $\varphi \in \Phi_0$ such that $\Sigma = \Sigma_\varphi$. This means that φ^S is induced by γ^S for all $S \in \mathfrak{S}_0(\mathcal{A})$. Since the S-ring \mathcal{A}_0 is separable, there exists an isomorphism γ_0 of the ring \mathcal{A}_0 to itself that induces φ . Without loss of generality we can assume that $1^{\gamma_0} = 1$. We claim that

$$(55) \quad x^{\gamma_0} = y^{\gamma'} \quad \text{for some } \gamma' \in \text{Aut}(\mathcal{A}_0), \quad 1^{\gamma'} = 1.$$

Then (53) holds for $\delta = \gamma_0(\gamma')^{-1}$ because $\gamma_0 \in \text{Aut}(\mathcal{B})$ by Lemma 3.7, and we are done.

Let us prove the claim. From Lemma 3.7 it follows that $\gamma_0 \in \text{Aut}(\mathcal{B})$. Therefore $x^{\gamma_0} \in X$ where X is the basic set of \mathcal{A} that contains x and y . Let $S \in \mathfrak{S}_0(\mathcal{A})$. Then the bijections γ_0^S and γ^S induce the same similarity φ^S of the S-ring $(\mathcal{A}_0)_S$. However, this S-ring equals $\mathbb{Z}S$ by statement (2) of Lemma 10.4. Therefore $\gamma_0^S = \gamma^S$. Thus the latter equality holds for all $S \in \mathfrak{S}_0(\mathcal{A})$.

Let now S be a section in the class $\widehat{\mathfrak{S}_0(\mathcal{A})}$ that has $\langle X \rangle / \text{rad}(X)$ as a subsection. Then $S_p \in \mathfrak{S}_0(\mathcal{A})$ for all primes p dividing $|S|$. By above this implies that

$$x_p^{\gamma_0} L = (x_p L)^{\gamma_0} = (x_p L)^\gamma = x_p^\gamma L = y_p L$$

where L is the denominator of S . On the other hand, the S-ring \mathcal{A}_S contains the tensor product of the S-rings \mathcal{A}_{S_p} , and hence the permutation $(\gamma_0)^S$ equals the product of its p -projections. Thus $x^{\gamma_0} L = y L$. By statement (2) of Theorem 10.11 this implies that the elements x^{γ_0} and y belong to the same basic set of \mathcal{A}_0 . Since this S-ring is schurian, there exists an automorphism $\gamma' \in \text{Aut}(\mathcal{A}_0)$ such that $x^{\gamma_0} = y^{\gamma'}$ and $1^{\gamma'} = 1$. The claim is proved. ■

Proof of Theorem 1.3. To prove the “only if” part suppose that the S-ring \mathcal{A} is schurian. Then by Theorem 5.8 there exists a group $\Gamma \in \mathcal{M}(\mathcal{A})$ that satisfies (54). Let $S \in \mathfrak{S}_0$. Then since $\mathfrak{S}_0 = \widehat{\mathfrak{S}_0(\mathcal{A})}$ (Corollary 10.10), we have $S_p \in \mathfrak{S}_0(\mathcal{A})$ for any prime p dividing $|S|$. Therefore due to (54) we conclude that

$$\Gamma^S \leq \prod_p \Gamma^{S_p} \leq \prod_p \text{Hol}(S_p) = \text{Hol}(S).$$

However, \mathcal{A}_S is the S-ring associated with the group Γ^S because \mathcal{A} is the S-ring associated with the group Γ . Thus the S-ring \mathcal{A}_S is cyclotomic and condition (1) is satisfied. To verify condition (2) we note that $\text{Aut}_{\mathcal{A}}(S) = \Gamma^S$. Therefore given $\sigma \in \text{Aut}_{\mathcal{A}}(S)$ one can find $\gamma \in \Gamma$ such that $1^\gamma = 1$ and $\gamma^S = \sigma$. Thus the element σ has a preimage $\Sigma = \{\gamma^S\}$ in the group $\text{Mult}(\mathcal{A})$.

To prove the “if part” suppose that for any section $S \in \mathfrak{S}_0$ the S-ring \mathcal{A}_S is cyclotomic and the restriction homomorphism from $\text{Mult}(\mathcal{A})$ to $\text{Aut}_{\mathcal{A}}(S)$ is surjective. Then by the first assumption the class \mathfrak{S}_0 is an \mathcal{A} -admissible (Corollary 10.10) and

$$(56) \quad \mathcal{A}_S = \text{Cyc}(\text{Aut}_{\mathcal{A}}(S), S) = (\mathbb{Z}S)^{\text{Aut}_{\mathcal{A}}(S)}$$

for all $S \in \mathfrak{S}_0$. Moreover, we claim that

$$(57) \quad (\mathbb{Z}S)^{\text{Aut}_{\mathcal{A}}(S)} = (\mathbb{Z}S)^{(\Phi_0)^S}$$

where $\Phi_0 = \Phi_0(\mathcal{A})$. Indeed, the left-hand side is obviously contained in the right-hand side. Suppose on the contrary that the reverse inclusion does not hold. Then there exists $\sigma \in \text{Aut}_{\mathcal{A}}(S)$ such that $\varphi_\sigma \notin (\Phi_0)^S$ where φ_σ is the similarity of $\mathbb{Z}S$ induced by σ . Moreover, by the surjectivity assumption there is a family $\Sigma \in \text{Mult}(\mathcal{A})$ the S -component of which equals σ . Since the class \mathfrak{S}_0 is an \mathcal{A} -admissible, Σ is an \mathfrak{S}_0 -multiplier of \mathcal{A} . So by Theorem 11.5 there exists a similarity $\varphi \in \Phi_0$ such that $\varphi_S = \varphi_\sigma$. This implies that $\varphi_\sigma \in (\Phi_0)^S$. Contradiction.

Set $\mathcal{A}' = (\mathcal{A}_0)^{\Phi_0}$. Then from (56) and (57) it follows that

$$(58) \quad (\mathcal{A}')_S = ((\mathcal{A}_0)^{\Phi_0})_S = ((\mathcal{A}_0)_S)^{(\Phi_0)^S} = (\mathbb{Z}S)^{(\Phi_0)^S} = \mathcal{A}_S$$

for all $S \in \mathfrak{S}_0$. By Theorem 1.2 to complete the proof it suffices to verify that $\mathcal{A} = \mathcal{A}'$. To do this we note that $\mathcal{A} \leq \mathcal{A}'$. So we only need to prove that $\mathcal{A} \geq \mathcal{A}'$, i.e. that

any basic set $X' \in \mathcal{S}(\mathcal{A}')$ belongs to $\mathcal{S}(\mathcal{A})$. However, obviously $S' = \langle X' \rangle / \text{rad}(X')$ is an \mathcal{A}_0 -section. Moreover, the extension $(\mathcal{A}_0)_{S'}$ of $(\mathcal{A}')_{S'}$ has trivial radical by Theorem 5.7. By Lemma 5.3 this implies that $S' \in \mathfrak{S}_0$. Denote by X the basic set of the S-ring $\mathcal{A}_{\langle X' \rangle}$ that contains X' . Then since $\mathcal{A}_{S'} = \mathcal{A}'_{S'}$ (see (58)), we have $\pi_{S'}(X) = \pi_{S'}(X')$. It follows that $X \subset X' \text{rad}(X') = X'$. Thus $X = X'$ as required. ■

13. PROOF OF THEOREM 1.4

13.1. Reduction to quasidense S-rings. Let \mathcal{A} be an S-ring over a cyclic group G . Following paper [9] a class C of projectively equivalent \mathcal{A} -sections is called *singular* if its rank is 2, its order is greater than 2 and it contains two sections $S = L_1/L_0$ and $T = U_1/U_0$ such that T is a multiple of S and the following two conditions are satisfied:

- (S1) \mathcal{A} is both the U_0/L_0 - and U_1/L_1 -wreath product,
- (S2) $\mathcal{A}_{U_1/L_0} = \mathcal{A}_{L_1/L_0} \otimes \mathcal{A}_{U_0/L_0}$.

By Lemma 6.2 of that paper S and T are necessarily the smallest and largest \mathcal{A} -sections of C . Moreover, from Theorem 4.6 there, it follows that any rank 2 class of composite order belonging to $\mathcal{P}(\mathcal{A})$ is singular; in particular, the S-ring \mathcal{A} is quasidense if and only if no class in $\mathcal{P}(\mathcal{A})$ is singular.

For an \mathcal{A} -section S we define the *S-extension* of \mathcal{A} to be the smallest S-ring $\mathcal{A}' \geq \mathcal{A}$ such that $\mathcal{A}'_S = \mathbb{Z}S$. From Theorem 4.2 it follows that \mathcal{A}' does not depend on the choice of S in the class $C \in \mathcal{P}(\mathcal{A})$ of sections projectively equivalent to S . Denote this S-ring by $E(\mathcal{A}, C)$.

The following lemma provides the reduction of Theorem 1.4 to the quasidense case, and will be used later only for a singular class C of composite order.

Lemma 13.1. *Let \mathcal{A} be a circulant S-ring, $C \in \mathcal{P}(\mathcal{A})$ a singular class and $\mathcal{A}' = E(\mathcal{A}, C)$. Then*

- (1) $\text{rk}(\mathcal{A}') > \text{rk}(\mathcal{A})$,
- (2) \mathcal{A}' is schurian if and only if \mathcal{A} is schurian.

Proof. Statement (1) follows from the fact that any singular class has rank 2 and order at least 3. To prove statement (2) denote by L_1/L_0 and U_1/U_0 the smallest and the largest \mathcal{A} -sections of C (see Theorem 5.4). First, we claim that \mathcal{A}' coincides with any extension \mathcal{B} of \mathcal{A} that satisfies the following conditions

- (E1) \mathcal{B} is both the U_0/L_0 - and U_1/L_1 -wreath product,
- (E2) $\mathcal{B}_{U_0} = \mathcal{A}_{U_0}$, $\mathcal{B}_{G \setminus L_1} = \mathcal{A}_{G \setminus L_1}$ and $\mathcal{B}_{U_1/L_0} = \mathbb{Z}S \otimes \mathcal{A}_{U_0/L_0}$

where $S = L_1/L_0$. Indeed, from condition (E2) it follows that $\mathcal{B}_S = \mathbb{Z}S$. So by the definition of \mathcal{A}' we have

$$\mathcal{A} \leq \mathcal{A}' \leq \mathcal{B} \quad \text{and} \quad (\mathcal{A}')_S = \mathbb{Z}S = \mathcal{B}_S.$$

This implies that conditions (E1) and (E2) are satisfied with \mathcal{B} replaced by \mathcal{A}' . It follows that

$$(59) \quad \mathcal{B}_{U_0} = \mathcal{A}'_{U_0}, \quad \mathcal{B}_{G \setminus L_1} = \mathcal{A}'_{G \setminus L_1}, \quad \mathcal{B}_{U_1/L_0} = \mathcal{A}'_{U_1/L_0},$$

and \mathcal{A}' and \mathcal{B} are the U_1/L_1 - and U_0/L_0 -wreath products. Therefore to check that $\mathcal{A}' = \mathcal{B}$ it suffices to verify that

$$\mathcal{A}'_{G/L_1} = \mathcal{B}_{G/L_1} \quad \text{and} \quad \mathcal{A}'_{U_1} = \mathcal{B}_{U_1}.$$

The former equality immediately follows from the second equality in (59). To verify the latter one, we observe that \mathcal{A}'_{U_1} and \mathcal{B}_{U_1} are the U_0/L_0 -wreath products. Thus the required statement follows from the other two equalities in (59).

To complete the proof we note that in terms of paper [9] every singular class is isolated (i.e. satisfies conditions (S1) and (S2)). Therefore by Lemma 6.5 of that paper the S-ring $\text{Ext}_C(\mathcal{A}, \mathbb{Z}S)$ defined there contains \mathcal{A} and satisfies conditions (E1) and (E2). By the above claim this implies that $\text{Ext}_C(\mathcal{A}, \mathbb{Z}S) = \mathcal{A}'$. Thus statement (2) follows from [9, Theorem 6.7]. ■

We recall that $\widehat{\mathcal{A}}$ is the S-ring dual to \mathcal{A} , and \widehat{C} is the class of projectively equivalent $\widehat{\mathcal{A}}$ -sections that is dual to a class $C \in \mathcal{P}(\mathcal{A})$ (Subsection 2.3).

Theorem 13.2. *Let \mathcal{A} be a circulant S-ring and C a singular class of \mathcal{A} . Then \widehat{C} is a singular class of $\widehat{\mathcal{A}}$ and the S-ring dual to $E(\mathcal{A}, C)$ coincides with $E(\widehat{\mathcal{A}}, \widehat{C})$.*

Proof. Let $S \in C$ be a section of rank 2 and order at least 3. Then the class \widehat{C} contains the section \widehat{S} . Since $|S| = |\widehat{S}|$ and $\text{rk}(\widehat{\mathcal{A}}_{\widehat{S}}) = \text{rk}(\mathcal{A}_S) = 2$ (Subsection 2.3), the class \widehat{C} has rank 2 and order greater than 2. Moreover, U_0^\perp/U_1^\perp and L_0^\perp/L_1^\perp are $\widehat{\mathcal{A}}$ -sections, and statement (1) of Lemma 4.4 implies that the former one is a multiple of the latter. Finally, by Theorem 2.3 the S-ring $\widehat{\mathcal{A}}$ satisfies conditions (S1) and (S2). Thus the class \widehat{C} is singular.

Let us prove that

$$(60) \quad E(\widehat{\mathcal{A}}, \widehat{C}) \leq \widehat{E(\mathcal{A}, C)}.$$

Indeed, since $E(\mathcal{A}, C) \geq \mathcal{A}$, the S-ring dual to $E(\mathcal{A}, C)$ contains $\widehat{\mathcal{A}}$. Moreover, since $E(\mathcal{A}, C)_S = \mathbb{Z}S$, the restriction of that ring to S^\perp equals $\mathbb{Z}S^\perp$. Thus (60) follows from the definition of $E(\widehat{\mathcal{A}}, \widehat{C})$. Next by duality, inclusion (60) also holds after interchanging \mathcal{A} and $\widehat{\mathcal{A}}$. Therefore

$$\widehat{E(\mathcal{A}, C)} \leq E(\widehat{\mathcal{A}}, \widehat{C}).$$

Due to inclusion (60) this completes the proof of the theorem. ■

Let us turn to the proof of Theorem 1.4. Let \mathcal{A} be a circulant S-ring. First, we observe that given a singular class $C \in \mathcal{P}(\mathcal{A})$ of composite order the S-ring $E(\mathcal{A}, C)$ is schurian if and only if so is \mathcal{A} (statement (2) of Lemma 13.1). Therefore by statement (1) of that lemma and by Theorem 13.2 without loss of generality we can assume that the S-ring \mathcal{A} has no singular classes of composite order, or equivalently that \mathcal{A} is a quasidense S-ring (Theorem 5.5).

13.2. Quasidense S-ring case. Given $\sigma \in \text{Aut}(G)$ we set $\widehat{\sigma}$ to be the automorphism of \widehat{G} taking a character χ to the character $\chi^{\widehat{\sigma}} : g \mapsto \chi(g^\sigma)$. Then $m(\sigma) = m(\widehat{\sigma})$ because

$$\chi^{\widehat{\sigma}}(g) = \chi(g^\sigma) = \chi(g^m) = \chi(g)^m, \quad g \in G,$$

where $m = m(\sigma)$. Moreover, this shows that $\widehat{\sigma^S} = \widehat{\sigma}^{\widehat{S}}$ for any section S of G .

Lemma 13.3. *Let \mathcal{A} be a quasidense circular S-ring. Then*

- (1) $\mathfrak{S}_0(\widehat{\mathcal{A}}) = \{\widehat{S} : S \in \mathfrak{S}_0(\mathcal{A})\},$
- (2) $\widehat{\mathcal{A}_0} = \widehat{\mathcal{A}}_0$ and $\mathfrak{S}_0(\widehat{\mathcal{A}}_0) = \{\widehat{S} : S \in \mathfrak{S}_0(\mathcal{A}_0)\},$

(3) $\text{Aut}_{\widehat{\mathcal{A}}}(\widehat{S}) = \{\widehat{\sigma} : \sigma \in \text{Aut}_{\mathcal{A}}(S)\}$ for all $S \in \mathfrak{S}(\mathcal{A})$.

Proof. Statement (1) follows from Theorem 5.10. From this statement and equivalence (1) \Leftrightarrow (4) of Theorem 8.3 it follows that a circulant S-ring is coset if so is its dual. Therefore the set of duals to coset S-rings containing \mathcal{A} coincides with the set of coset S-rings containing $\widehat{\mathcal{A}}$. Thus $\widehat{\widehat{\mathcal{A}}_0} = \widehat{\mathcal{A}}_0$ by the definition of the coset closure. The second part of statement (2) follows from the first one and statement (1).

To prove statement (3) let $S \in \mathfrak{S}(\mathcal{A})$. Then the group $\text{Aut}_{\mathcal{A}}(S)$ equals the largest group $K \leq \text{Aut}(S)$ for which $\text{Cyc}(K, S) \geq \mathcal{A}_S$. Since the dual to the S-ring $\text{Cyc}(K, S)$ equals $\text{Cyc}(\widehat{K}, \widehat{S})$ where $\widehat{K} = \{\widehat{\sigma} : \sigma \in K\}$ (Theorem 2.2), the group \widehat{K} is the largest subgroup of $\text{Aut}(\widehat{S})$ for which $\text{Cyc}(\widehat{K}, \widehat{S}) \geq \widehat{\mathcal{A}}_{\widehat{S}}$. So $\text{Aut}_{\widehat{\mathcal{A}}}(\widehat{S}) = \widehat{K}$ and we are done. ■

To complete the proof of Theorem 1.4 let \mathcal{A} be a schurian quasidense circulant S-ring. By duality we only have to prove that the S-ring $\widehat{\mathcal{A}}$ is schurian. However, the latter ring is quasidense by statement (2) of Theorem 5.9. Therefore by Theorem 1.3 and statement (3) of Corollary 10.10 it suffices to verify that both of its conditions are satisfied for the S-ring $\widehat{\mathcal{A}}$ and sections belonging to $\mathfrak{S}_0(\widehat{\mathcal{A}}_0)$.

Let $T \in \mathfrak{S}_0(\widehat{\mathcal{A}}_0)$. Then by statement (2) of Lemma 13.3 there exists a section $S \in \mathfrak{S}_0(\mathcal{A}_0)$ such that $T = \widehat{S}$. Since \mathcal{A} is schurian, the S-ring \mathcal{A}_S is cyclotomic (Theorem 1.3). Therefore the S-ring $\widehat{\mathcal{A}}_T = \widehat{\mathcal{A}}_S$ is cyclotomic by Theorem 2.2. Thus condition (1) is satisfied.

To verify condition (2), let $\tau \in \text{Aut}_{\widehat{\mathcal{A}}}(T)$. Since $T \in \mathfrak{S}(\widehat{\mathcal{A}})$ (Theorem 10.7), statement (3) of Lemma 13.3 implies that there exists $\sigma \in \text{Aut}_{\mathcal{A}}(S)$ such that $\tau = \widehat{\sigma}$. Moreover, since \mathcal{A} is schurian there exists an element $\Sigma = \{\sigma_{S'}\}$ of the group $\text{Mult}(\mathcal{A})$ such that $\sigma_S = \sigma$ (Theorem 1.3). Set

$$\widehat{\Sigma} = \{\sigma_{T'}\}_{T' \in \mathfrak{S}_0(\widehat{\mathcal{A}}_0)}$$

where $\sigma_{T'} = \widehat{\sigma_{S'}}$ with S' defined by $\widehat{S'} = T'$ (see statement (2) of Lemma 13.3). Since $\widehat{\widehat{S}} = \widehat{\sigma} = \tau$, it suffices to verify that $\widehat{\Sigma} \in \text{Mult}(\widehat{\mathcal{A}})$. By the remark after Definition 11.1 we have to verify that conditions (M1) and (M2) of that definition are satisfied for the class $\mathfrak{S}_0(\widehat{\mathcal{A}}_0)$ and family $\widehat{\Sigma}$.

Let $\tau_1 \in \text{Aut}_{\widehat{\mathcal{A}}}(T_1)$ where $T_1 \in \mathfrak{S}_0(\widehat{\mathcal{A}}_0)$. Then since Σ is a multiplier, for every subsection T_2 of T_1 we have

$$(\tau_1)^{T_2} = (\widehat{\sigma_1})^{\widehat{S_2}} = (\widehat{\sigma_1})^{S_2} = \widehat{\sigma_2} = \tau_2$$

where S_i is such that $T_i = \widehat{S_i}$ and $\sigma_i \in \text{Aut}_{\mathcal{A}}(S_i)$ is such that $\tau_i = \widehat{\sigma_i}$, $i = 1, 2$. Thus condition (M1) is satisfied. To verify condition (M2) let now $T_2 \sim T_1$. Then by statement (1) of Lemma 4.4 we have $S_1 \sim S_2$. Since Σ is a multiplier, this implies that $m(\sigma_1) = m(\sigma_2)$. Thus

$$m(\tau_1) = m(\widehat{\sigma_1}) = m(\sigma_1) = m(\sigma_2) = m(\widehat{\sigma_2}) = m(\tau_2)$$

as required.

REFERENCES

- [1] E. Bannai, T. Ito, *Algebraic combinatorics. I*, Benjamin/Cummings, Menlo Park, CA, 1984.
- [2] J. D. Dixon, B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics, No. 163, Springer-Verlag New York, 1996.
- [3] S. Evdokimov, I. Ponomarenko, *On a family of Schur rings over a finite cyclic group*, Algebra and Analysis, **13** (2001), 3, 139–154.
- [4] S. Evdokimov, I. Ponomarenko, *Characterization of cyclotomic schemes and normal Schur rings over a cyclic group*, Algebra and Analysis, **14** (2002), 2, 11–55.
- [5] S. Evdokimov, I. Ponomarenko, *Recognizing and isomorphism testing circulant graphs in polynomial time*, Algebra and Analysis, **15** (2003), 6, 1–34.
- [6] S. Evdokimov, I. Ponomarenko, *A new look at the Burnside-Schur theorem*, Bulletin of the London Mathematical Society, **37** (2005), 535–546.
- [7] S. Evdokimov, I. Ponomarenko, *Permutation group approach to association schemes*, European Journal of Combinatorics, **30** (2009), 6, 1456–1476.
- [8] S. Evdokimov, I. Ponomarenko, *Schur rings over a Galois ring of odd characteristic*, Journal of Combinatorial Theory, **A117** (2010), 827–841.
- [9] S. Evdokimov, I. Ponomarenko, *Schurity of S -rings over a cyclic group and generalized wreath product of permutation groups*, Algebra and Analysis, **24** (2012), 3, 84–127.
- [10] S. Evdokimov, I. Kovács, I. Ponomarenko, *Characterization of cyclic Schur groups*, Algebra and Analysis, **25** (2013), 5, Algebra and Analysis, 61–85.
- [11] S. Evdokimov, I. Ponomarenko, *Schur rings over a product of Galois rings*, Beitr Algebra Geom., **55** (2014), 105–138.
- [12] M. Klin, R. Pöschel, *The König problem, the isomorphism problem for cyclic graphs and the method of Schur rings*, In: “Algebraic Methods in Graph Theory, Szeged, 1978”, Colloq. Math. Soc. János Bolyai, Vol. 25, North-Holland, Amsterdam (1981), 405–434.
- [13] K. H. Leung, S. H. Man, *On Schur Rings over Cyclic Groups, II*, J. Algebra, **183** (1996), 273–285.
- [14] E. M. Luks, *Permutation groups and polynomial-time computation*, American Mathematical Society. DIMACS, Ser. Discrete Math. Theor. Comput. Sci. 11, Providence, RI: American Mathematical Society, **1993**, 139–175.
- [15] M. Muzychuk, *A wedge product of association schemes*, European J. Combin., **30** (2009), 705–715.
- [16] R. Schmidt, *Subgroup lattices of groups*, De Gruyter Expositions in Mathematics. 14. Berlin: Walter de Gruyter (1994).
- [17] H. Wielandt, *Finite permutation groups*, Academic press, New York - London, 1964.
- [18] H. Wielandt, *Permutation groups through invariant relations and invariant functions*, Lect. Notes Dept. Math. Ohio St. Univ., Columbus, 1969.

ST. PETERSBURG DEPARTMENT OF STEKLOV INSTITUTE OF MATHEMATICS, RUSSIA
E-mail address: `evdokim@pdmi.ras.ru`

ST. PETERSBURG DEPARTMENT OF STEKLOV INSTITUTE OF MATHEMATICS, RUSSIA
E-mail address: `inp@pdmi.ras.ru`